MANY - SORTED RESOLUTION FOR WELL - FORMED FORMULAE WITH EQUALITY

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Dedicated

to

My Parents

CERTIFICATE

This is to certify that the thesis work entitled **Many-Sorted Resolution for Well-Formed Formulae with Equality** by Mr.K.S.H.S.R.Bhatta has been carried out under my supervision and has not been submitted elsewhere for a degree.

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ABSTRACT

Several *new* proof procedures, namely, *Non-Clausal Resolution by Unification and Equality (NCRUE), many-sorted NCRUE, Non-Clausal resolution with quantifiers in place (WFF-resolution), many-sorted WFF-resolution, NCRUE with quantifiers in place (WRUE),* and *many-sorted WRUE,* are described in this thesis.

The existing rule set for converting an unsorted set of well-formed formulae into a many-sorted set is augmented with additional rules, one rule for obtaining more sort information, two rules based on equality and two rules for sorting terms quantified within an equivalence in the respective proof procedures. The existing algorithm for obtaining sort information is appropriately modified to include application of the additional rules. Also given is an algorithm for obtaining dependencies of existential variables in a wff without paraphrasing any connective in terms of the others.

All the introduced proof procedures together with the required algorithms are proved to be sound and complete. Examples are also given to illustrate the application of proof procedures.

NOTATION

Symbol	Meaning
$\sim$	Negation
$\vee$	Disjunction
$\&$	Conjunction
$\longrightarrow$	Implication
$\longleftrightarrow$	Equivalence (or bi-implication)
$U\langle F\rangle$	wff U contains sub-wff F
$U\langle F^{+}\rangle$	wff U contains positive sub-wff F
$U\langle F^{-}\rangle$	wff U contains negative sub-wff F
$U\langle t_1 = t_2\rangle$	wff U contains equality $t_1 = t_2$
$U\langle t_1 =^{+} t_2\rangle$	wff U contains positive equality $t_1 = t_2$
$U\langle t_1 =^{-} t_2\rangle$	wff U contains negative equality $t_1 = t_2$

Other notations, if not standard, are explained where they are first used.

CONTENTS

Chapter 1

INTRODUCTION

1.1 Automation of Reasoning

The process of *reasoning,* formally involves a series of applications of logical rules of inference on a set of facts to arrive at a conclusion. For easy mechanization the necessary rules of inference must be simple and few.

The language through which the problems are presented to a theorem prover is the *First-Order Predicate Calculus.* A problem consists of *facts,* also known as *axioms* or *premises* and a *conclusion,* also known as a *theorem.* Basically there are two different approaches for solving a given problem. One of them is to start with the set of axioms and arrive at a conclusion, the approach which closely resembles the intuitive reasoning of humans and is called *natural deduction.* The other is the well known *reductio ad absurdum* in which we start with a negation of the conclusion and derive a contradiction. The latter approach, more commonly known as a *refutation procedure,* is one on which most of the working theorem provers are based.

Automated reasoning through resolution would be more

tractable if we can do the following:

(i) Retain the original form of the formula which will make it easier for humans to guide the theorem prover,

(ii) introduce domain information which would weed out useless inferences, and

(iii) include equality and other special relations, which are used in a wide variety of domains, in an efficient way in the prover.

1.2 Syntactic Structure in Reasoning

Keeping in view of the idea to retain the syntactic structure of the wffs in resolution, we shall briefly look at resolution principle, non-clausal resolution and WFF-resolution (NC-resolution on wffs with quantifiers in place) in this section.

1.2.1 Resolution Principle

The *resolution principle* is a single rule of inference that can be used to build up a refutation proof procedure [Rob 65] from a given set of clauses.

Given the First-Order Logic (FOL) formulation of a problem, the first step is to convert the axioms and the negated theorem into a set of clauses [ChL 73], [Lov 78] or [Qui 61]. A *clause* is a disjunction of literals, where a *literal* is a positive or negative atom.

Let us for illustration, consider the following problem from [ChL 73].

Fact/Premise/Axiom :

Everyone who saves money earns interest.

Conclusion/Theorem :

If there is no interest, then nobody saves money.

Let the predicate symbols be defined as below:

S(x, y) : x saves y,

M(x) : x is money,

I(x) : x is interest,

E(x, y) : x earns y.

In FOL formulation of the above problem, the premise is

$$S_1: (\forall x)\ [(\exists y)\ (S(x, y)\ \&\ M(y)) \dashrightarrow (\exists y)\ (I(y)\ \&\ E(x, y))]$$

and the negated conclusion is

$$S_2: \sim[\sim(\exists x)\ I(x) \dashrightarrow (\forall x)\ (\forall y)\ (S(x, y) \dashrightarrow \sim M(y))]$$

Converting these sentences into clauses, we have

C_1: $\sim S(x, y)\ V \sim M(y)\ V\ I(f(x))$

C_2: $\sim S(x, y)\ V \sim M(y)\ V\ E(x, f(x))$

C_3: $\sim I(z)$

C_4: $S(a, b)$

C_5: $M(b)$

Resolution can be performed on two clauses with complementary predicates, if there is some substitution that makes these atoms identical. The process of finding such substitution is called *unification*.

For example, in the above set, the atom I is complementary in C_1, C_3 and the substitution $\{f(x)/z\}$ makes them identical, and hence can be resolved to give the resolvent

$$C_1 + C_3 \dashrightarrow R_1\text{: } \sim S(x, y) \vee \sim H(y)$$

A *resolvent* is a disjunction of the two clauses with resolved-upon atoms (positive in one, and negative in the other) removed.

Now, we can resolve R_1, C_4 with substitution $\{a/x, b/y\}$ to give

$$C_4 + R_1 \dashrightarrow R_2\text{: } \sim H(b)$$

Finally, resolving R_2, C_5, we get a *null clause* as the resolvent.

$$C_5 + R_2 \dashrightarrow R_3\text{: } \square$$

It is known that for all satisfiable sets of clauses, the union of such a set with the set of its resolvents is

also *satisfiable*. But a $\square$ is *unsatisfiable*, and hence the given set of clauses should also be unsatisfiable. So whenever the axioms are true, the negated theorem is false, or the theorem follows from the axioms.

1.2.2 Non-Clausal Resolution

The above refutation proof would be easier and more understandable if the translation of the sentences to clauses could be avoided.

A resolution rule called *NC-resolution* has been introduced by Murray [Mur 82] which operates on the sentences as they are, except that they are made quantifier free.

NC-resolution can be performed on two quantifier free wffs with complementary sub-wffs (need not be just predicates, but can also be wffs) by making appropriate substitutions. An *NC-resolvent* is obtained by substituting and simplifying, *FALSE* in place of the positive sub-wff and *TRUE* in place of the negative sub-wff, and then taking a disjunction of the two sentences.

The quantifier free version of the above problem is

S_1: $[(S(x, y) \& M(y)) \dashrightarrow (I(f(x)) \& E(x, f(x)))]$

S_2: $\sim[\sim I(z) \dashrightarrow (S(a, b) \dashrightarrow \sim M(b))]$

In the above set of wffs, since the atoms with I occur positively in S_1 and negatively in S_2, they can be

NC-resolved with the substitution $\{f(x)/z\}$ to give

$$S_1 + S_2 \text{ ---> } NCR_1\text{:}$$
$$[(S(x, y) \,\&\, M(y)) \text{ --> } (FALSE \,\&\, E(x, f(x))]$$
$$V \sim[\sim TRUE \text{ --> } (S(a, b) \text{ --> } \sim M(b))]$$

which simplifies to

$$\sim[S(x, y) \,\&\, M(y)].$$

Now, NCR_1 and S_2 can be resolved with the substitution $\{a/x, b/y\}$ to give

$$S_2 + NCR_1 \text{ ---> } NCR_2\text{: } \sim M(b)$$

Now, NCR_2 can be resolved with S_2 to give a *FALSE.*

$$S_2 + NCR_2 \text{ ---> } NCR_3\text{: } FALSE.$$

Thus, a derivation of *FALSE* by NC-resolution, analogous to a derivation of □ by resolution, represents unsatisfiability of the given set.

In fact, if we can identify that NCR_1 is identical to $(S(a, b) \text{ --> } \sim M(b))$ in S_2 under the substitution $\{a/x, b/y\}$, we can NC-resolve on these sub-wffs, as they are com-

plementary. Thus, the resolvent would be

$$S_2 + NCR_1 \text{ ---> } NCR_2\text{: } \sim[TRUE] \vee \sim[\sim I(z) \text{ --> } \sim FALSE]$$

which simplifies to

$$FALSE.$$

Thus, the length of the refutation gets reduced when we resolve on sub-wffs in NC-resolution.

1.2.3 WFF-resolution

A further step would be when we can resolve on wffs as they are with quantifiers in place. A resolution rule called *WFF-resolution* (resolution on wffs with quantifiers in place) is being introduced in this thesis (Chapter 6), by which we can resolve on sentences as they are.

WFF-resolution is NC-resolution with quantifiers in place. Here, Skolem constants and functions are captured by *functional dependencies.* In addition to truth-functional simplification the generation of the resolvent requires *merging* of quantifiers.

The set of wffs for the previous example with quantifiers in place and functional dependencies substituted for existential variables (the method for obtaining dependencies is described in Chapter 6) would be

$$S_1: (\forall x)\ [(\exists y)\ (S(x, y)\ \&\ M(y))$$
$$\quad ---\!\!> (\exists z(x, y))\ (I(z)\ \&\ E(x, z))]$$

$$S_2: \sim[\sim(\exists u)\ I(u)\ ---\!\!> (\forall v(u))\ (\forall w(u))\ (S(v, w)\ ---\!\!> \sim M(w))]$$

No functional dependencies are obtained for y in S_1, because effectively it is universally quantified. Similarly, for u in S_2. Since v, w are existentially quantified in S_2, their dependency is (u). Actually, these were substituted only by Skolem constants rather than functions in making the wff quantifier free, because $\&$ is commutative. Hence, in such a case, using functions or constants does not affect the (un)satisfiability.

With quantifiers in place also, we can observe that the atoms with I occur positively in S_1 and negatively in S_2, and the terms to be unified are $(\exists z(x, y))$ and $(\forall u)$. With the substitution $\{(\exists z(x, y))/(\forall u)\}$, we can resolve S_1 and S_2 to give

$$S_1 + S_2 ---\!\!> R_1: (\forall x)\ \sim[(\exists y)\ (S(x, y)\ \&\ M(y))]$$

Now, R_1 can be resolved with S_2 on atoms S with the substitution $\{(\exists v(u))/(\forall x), (\exists w(u))/(\forall y)\}$ to give

$$S_2 + R_1 ---\!\!> R_2: (\forall u)\ (\exists w(u))\ \sim M(w)$$

Thus, forming a resolvent in WFF-resolution may involve

dropping quantifiers and adding quantifiers. For example, in the above resolution, $(\forall x)$, $(\forall y)$ have been dropped and $(\forall u)$, $(\exists w(u))$ have been added.

Finally, R_2 and S_2 can be resolved with empty substitution to give *FALSE*.

$S_2 + R_2$ ---> R_3: *FALSE*.

Similar to NC-resolution, a derivation of *FALSE* proves the unsatisfiability of the given set of wffs.

An algorithm for obtaining dependencies for existential variables and a refutation procedure, *WFF-resolution*, is given in Chapter 6.

1.3 Semantics in Reasoning

In a refutation procedure often a large number of resolutions are used up simply to remove *unary* literals which describe the *type* or *sort* of the term. By extending resolution to a many-sorted logic by suitably constraining unification we can considerably shorten the lengths of proofs.

Many-sorted logic has been extended to resolution and paramodulation by Walther [Wal 82] and to NC-resolution by Jaya Raghavendra [Jay 87]. In chapter 7, we describe a many-sorted resolution procedure for wffs.

Let us for illustration, consider the following problem.

Premises:

S_1: All men are mortal,

S_2: Socrates is a man,

Conclusion:

S_3: Socrates is mortal.

With two predicates, namely, *MAN(x)* and *MORTAL(x)* denoting *"x is a man"* and *"x is mortal"* respectively, FOL translation of the above problem (with negated conclusion) is

S_1: $(\forall x)\ [MAN(x) \longrightarrow MORTAL(x)]$

S_2: $MAN(Socrates)$

S_3: $\sim MORTAL(Socrates)$.

The variable x in

S_1: $(\forall x)\ [MAN(x) \longrightarrow MORTAL(x)]$

has an unrestrained domain. It can stand for any object in the domain of all objects. But it is useless to perform inferences (resolutions) where x is substituted by objects which are not from the category *MAN*. This is because such

resolvents should indeed be satisfiable, as S_1 is true for all *x's* which are not *MAN*. So, essentially we can treat S_1 as equivalent to

$$MS_1: \forall x{:}MAN,\ MORTAL(x)$$

which states that *MORTAL(x)* is true for all *x* which are of the sort *MAN*. Similarly S_2 states that *Socrates* is of the sort *MAN*. Thus, the sorted version of the above example would be

$$MS_1: \forall x{:}MAN,\ MORTAL(x)$$

$$MS_2: Socrates{:}MAN$$

$$MS_3: \sim MORTAL(Socrates).$$

Now, MS_1 can be resolved with MS_3, as *Socrates* an element of the sort *MAN* can be substituted for the *x* which is also of the same sort, thus proving the theorem.

Such a process where the sortedness of the problem is used is called *Many-sorted resolution*. Its main advantage is the reduction in proof depth because of constrained unification. Unsorted refutation of the above problem requires 2 resolutions.

1.4 Equality in Reasoning

Equality is the most important special relation which has been incorporated into theorem provers. *Paramodulation*

[WoR 69] and *E-resolution* [Mor 69] were essentially developed for this purpose. However, both these methods do not really solve the problem. Paramodulation is intractable and E-resolution is incomplete.

A successful attempt (i.e. dispensing with the explicit use of equality axioms, and yet be able to reason about equality) has been made by Digricoli and Harrison [DiH 86]. Two rules of inference, *RUE (Resolution by Unification and Equality)* and *NRF (Negative Reflective Function)*, have been introduced in [DiH 86] that are primarily based on the resolution principle.

The rules of inference can be stated as below.

RUE rule of inference:

The *RUE resolvent* of $A \vee P(s_1,...,s_n)$, $B \vee \sim P(t_1,...,t_n)$ is $A\theta \vee B\theta \vee D$, where θ is a substitution and D is a disjunction of inequalities specified by a disagreement set of $P(s_1,...,s_n)\theta$, $\sim P(t_1,...,t_n)\theta$.

NRF rule of inference:

The *NRF resolvent* of $A \vee t_1 \neq t_2$ is $A\theta \vee D$, where θ is a substitution and D is a disjunction of inequalities specified by a disagreement set of $t_1\theta$, $t_2\theta$.

A *disagreement set* of two complementary predicates is a set of pairs of nonidentical terms. For example, a disagreement set of $P(a, x, b)$ and $\sim P(c, d, e)$ is $\{a:c, b:e\}$, since d can substitute for x.

As an example, let us consider the following set of clauses from [DiH 86]

S_1: $P(f(c))$

S_2: $\sim P(g(e)) \vee h(i(j(a), x, j(x))) \neq h(i(j(b), c, j(a)))$

S_3: $a = b$

S_4: $b = c$

S_5: $f(a) = g(e)$

and its refutation is

$S_1 + S_2 \longrightarrow R_1$:

$f(c) \neq g(e) \vee h(i(j(a), x, j(x))) \neq h(i(j(b), c, j(a)))$

$R_1 + S_5 \longrightarrow R_2$:

$a \neq c \vee h(i(j(a), x, j(x))) \neq h(i(j(b), c, j(a)))$

Now, applying NRF to the inequality $h(i(j(a), x, j(x))) \neq h(i(j(b), c, j(a)))$ with the substitution $\{c/x\}$,

$R_2 \longrightarrow_{NRF} R_3$: $a \neq c \vee a \neq b$;*merging into* $a \neq c$

and continuing RUE resolution,

$R_3 + S_4 \longrightarrow R_4$: $a \neq b$;*merging into* $a \neq b$

$R_4 + S_3 \longrightarrow R_5$: $\square$

If we look at the data presented in [DiH 86] the

refutation using equality axioms with unification resolution takes 16 steps, that with hyperparamodulation takes 11 steps, and unrestricted paramodulation 8 steps.

We extend equality-based resolution to NC-resolution as well as WFF-resolution. Equality-based NC-resolution (NCRUE resolution) is described in Chapter 4, and its extension to WFF-resolution in Chapter 8. NCRUE resolution is also extended to include many-sorted resolution in Chapter 5.

In Chapter 9 we put everything together and describe a many-sorted, equality-based WFF-resolution procedure.

1.5 Overview of the Thesis

An overview of the full thesis is as follows:

Chapter 2 describes NC-resolution and Chapter 3 the Many-sorted NC-resolution, where the sort information is generated automatically.

An equality-based NC-resolution called *Non-clausal Resolution by Unification and Equality (NCRUE)*, an extension of RUE resolution to non-clausal case, is discussed in Chapter 4. The soundness and completeness proofs for such a proof procedure are also given in Chapter 4.

Chapter 5 discusses a Many-sorted and Equality-based NC-resolution system, called *Many-sorted NCRUE resolution (MNCRUE).* It includes a discussion on how equality affects sorts and also proves its soundness and completeness.

In Chapter 6, a resolution system for well-formed for-

mulae with quantifiers in place, called *WFF-resolution* is introduced where functional dependencies for existential variables are obtained automatically without modifying the structure of the sentences. The soundness and completeness of WFF-resolution is proved in the chapter.

Chapter 7 describes a *Many-sorted WFF-resolution* proof procedure and proves its soundness and completeness.

Chapter 8 discusses NCRUE resolution in the light of having quantifiers in place, i.e. Equality-based WFF-resolution, called *WFF-Resolution by Unification and Equality (WRUE)*. The soundness and completeness of WRUE is also established.

WRUE resolution is extended to many-sorted theory in Chapter 9. Such a resolution, called *Many-sorted WRUE (MWRUE)* resolution is explained and proved to be sound and complete in Chapter 9.

Examples are given in Chapter 10 to illustrate the application of all proof procedures introduced in this thesis.

The last chapter, Chapter 11, concludes and mentions some directions in which the work can be extended.

Chapter 2

NON-CLAUSAL RESOLUTION

The resolution principle put forth by J.A.Robinson [Rob 65] requires that the sentences be converted to clausal form. The clausal form has many disadvantages [Sti 85].

(1) The intuition behind selecting appropriate connectives in expressing the problem is lost in the conversion.

(2) A sentence may breakup into a large number of clauses, resulting in substantial redundancy in the resolution search space.

(3) The clausal form is difficult to read and is not human oriented. Guiding the theorem prover and understanding the proofs become difficult.

The non-clausal resolution principle called *NC-resolution* proposed by Murray.N.V [Mur 82] removes the above disadvantages. A variation of NC-resolution was used for program synthesis by Manna and Waldinger [MaW 80]. NC-resolution can be performed on unifiable sub-sentences in two quantifier free sentences of First-Order Logic. The sentences may have the entire set of connectives including *implications* and *equivalences*. The completeness of the NC-resolution has also been proved in [Mur 82].

2.1 Introduction

The reader is assumed to be familiar with the notation and terminology of resolution and unification [Rob 65]. To provide continuity we define a few key terms. An *atom* is an n-place predicate symbol with terms as arguments. $P(x)$, $Q(a, f(a))$ are all atoms. A *literal* is an atom or the negation of an atom. A *clause* is a disjunction of literals. A *sentence* or a *well-formed formula (wff)* is formally defined as follows:

(i) An atom is a wff.

(ii) If F and G are wffs then so are $\sim F$, $(F \,\&\, G)$, $(F \vee G)$, $(F \longrightarrow G)$ and $(F \longleftrightarrow G)$.

(iii) If F is a wff then so are $(\forall x)F$ and $(\exists x)F$.

(iv) Wffs are generated only by a finite number of applications of (i), (ii) and (iii).

The parentheses are either retained or dropped, to enhance readability of wffs. We write $F\langle x\rangle$ to indicate that x occurs in the wff F; x may be a term, an atom or a wff. If x is a wff and $F\langle x\rangle$, then x is said to be a *sub-wff*.

In clausal resolution we resolve on complementary literals in two clauses. Similarly, in NC-resolution we resolve on unifiable sub-wffs, with opposite polarities in the two sentences. If F is a sub-wff in the wff S, then the parity of the number of explicit and implicit negations in whose scope F appears in S, gives the polarity of F

in S (positive if even, negative if odd). For example the literal $P(x)$ occurs negative in $P(x) \dashrightarrow Q(x)$, but positive in $(P(x) \dashrightarrow Q(x)) \dashrightarrow R(x)$. Formally we define *polarity* of the constituents of a wff as follows:

Let F, S_1 and S_2 be wffs.

(1) If F is positive (negative) in S_1, then it is negative (positive) in $\sim S_1$ and $(S_1 \dashrightarrow S_2)$,

(2) If F is positive (negative) in S_1, then it is positive (negative) in $(S_1 \& S_2)$, $(S_1 \vee S_2)$ and $(S_2 \dashrightarrow S_1)$,

(3) If F occurs in S_1, then it is both positive and negative in $(S_1 \langle\text{--}\rangle S_2)$.

In clausal resolution, the resolved-upon literals are deleted and the remaining literals are disjoined to form the resolvent. Similarly, in NC-resolution, all occurrences of the resolved-upon sub-wffs are replaced by, *FALSE* in place of the positive occurrence and *TRUE* in place of the negative occurrence. The resulting wffs are disjoined and simplified by truth functional reductions to eliminate embedded occurrences of *TRUE* and *FALSE*.

To illustrate, let us consider the following sentences:

S_1: $MAN(x) \dashrightarrow HUMAN(x)$

S_2: $MAN(Socrates)$.

The atoms with MAN in S_1 and S_2 can be unified, and

occur with opposite polarities (negative in S_1 and positive in S_2). So the NC-resolution on them gives

$$TRUE \rightarrow HUMAN(Socrates) \lor FALSE$$

which simplifies to

$$HUMAN(Socrates).$$

An NC-resolution derivation of *FALSE* from a set of wffs demonstrates the unsatisfiability of the set of wffs. Non-clausal resolution is thus like clausal resolution, a refutation procedure. Also it can be observed that NC-resolution reduces to clausal resolution when the wffs are restricted to be clauses.

2.2 Structural Equivalence

An important advantage of non-clausal resolution is that it allows resolution not only on literals but also on sub-wffs.

Consider the example:

Everyone who eats some plant is a Herbivore.

All wolves eat some plant.

Therefore, Wolfy is a Herbivore.

Its FOL translation is

$$S_1: (\forall x) [[(\exists y) (P(y) \& EAT(x, y))] \rightarrow H(x)]$$

S_2: $(\forall x)\ [W(x) \longrightarrow (\exists y)\ [P(y)\ \&\ EAT(x, y)]]$

S_3: $W(Wolfy)$

S_4: $\sim H(Wolfy)$.

A simple observation tells us that a resolution on the emboldened portions of S_1 and S_2, is better than a sequence of resolutions where the atoms with P and EAT are removed one at a time. A clausal deduction of $FALSE$ is given below for comparison.

Clausal solution:

The set of clauses:

C_1: $\sim P(y_1)\ V\ \sim EAT(x_1, y_1)\ V\ H(x_1)$

C_2: $\sim W(x_2)\ V\ P(f(x_2))$

C_3: $\sim W(x_3)\ V\ EAT(x_3, f(x_3))$

C_4: $W(Wolfy)$

C_5: $\sim H(Wolfy)$

The sequence of resolutions:

$C_4 + C_3 \longrightarrow Rs_1$: $EAT(Wolfy, f(Wolfy))$

$C_5 + C_1 \longrightarrow Rs_2$: $\sim P(y_1)\ V\ \sim EAT(Wolfy, y_1)$

$Rs_1 + Rs_2 \longrightarrow Rs_3$: $\sim P(f(Wolfy))$

$Rs_3 + C_2 \longrightarrow Rs_4$: $\sim W(Wolfy)$

$Rs_4 + C_4 \longrightarrow Rs_5$: $\square$

The theorem is proved.

Now the solution of the above example using NC-resolution, where the resolved-upon sub-wffs may be non-atomic, is given below.

$$S_1 + S_2 \text{ ---> } R_1 : H(x) \vee \sim H(x)$$
$$R_1 + S_3 \text{ ---> } R_2 : H(Wolfy)$$
$$R_2 + S_4 \text{ ---> } R_3 : FALSE.$$

The refutation process is faster when the resolved-upon sub-wffs are non-atomic. This corresponds to taking longer reasoning steps. It is easy to identify that emboldened sub-wffs are resolvable upon. The sentence S_1' below, which is a logical equivalent of S_1, hides this fact.

$$S_1' : (\forall x)\ (\forall y)\ [P(x) \text{ --> } \sim EAT(x, y)] \vee H(x)$$

The concept of structural equivalence helps us recognize when two sub-wffs are resolvable.

2.2.1 Definition (Strict S-equivalence)

Two wffs F, G are said to be *strictly S-equivalent* in all of the following cases.

(a) F and G are atoms, and have the same predicate symbol,

(b) $F = \sim F_1$, $G = \sim G_1$ and F_1, G_1 are strictly S-

equivalent,

(c) $F = (F_1\ b\ F_2)$, $G = (G_1\ b\ G_2)$ for any binary connective b, and F_1, G_1 and F_2, G_2 are strictly S-equivalent,

(d) $F = [Qx]\ F_1$, $G = [Qx]\ G_1$ where Q is some quantifier and x is some variable, and F_1, G_1 are strictly S-equivalent. $[Qx]$ represents optional quantifier.

2.2.2 Definition (Reducible S-equivalence)

If there exists a finite sequence of transformations, from the following set, which when applied to a wff F produces F_1 and/or a wff G produces G_1 (referred to the S-equivalent form) such that F_1, G, or F, G_1, or F_1, G_1 are strictly S-equivalent, then F, G are *reducibly S-equivalent.*

The set of transformation rules represented as equivalences are:

1. $(\sim(\sim A)) \longleftrightarrow A$
2. $(A \longrightarrow B) \longleftrightarrow ((\sim A)\ V\ B)$
3. $(A \longleftrightarrow B) \longleftrightarrow (A \longrightarrow B)\ \&\ (B \longrightarrow A)$
4. $(\sim (A\ V\ B)) \longleftrightarrow ((\sim A)\ \&\ (\sim B))$
5. $(\sim (A\ \&\ B)) \longleftrightarrow ((\sim A)\ V\ (\sim B))$
6. $(A\ \&\ B) \longleftrightarrow (B\ \&\ A)$
7. $(A\ V\ B) \longleftrightarrow (B\ V\ A)$
8. $(A\ \&\ (B\ \&\ C)) \longleftrightarrow ((A\ \&\ B)\ \&\ C)$
9. $(A\ V\ (B\ V\ C)) \longleftrightarrow ((A\ V\ B)\ V\ C)$

10. (A V (B & C)) <--> (A V B) & (A V C)

11. (A & (B V C)) <--> (A & B) V (A & C)

2.2.3 Definition (S-equivalence)

Two wffs are said to be *S-equivalent* if they are either strictly S-equivalent or reducibly S-equivalent.

From the above definition we can find that emboldened portions in S_1 and S_2 are strictly S-equivalent, and those in S_1 and S_1' are reducibly S-equivalent, and in both cases they are S-equivalent. We now use S-equivalence to determine the *complementarity* of two sub-wffs.

2.2.4 Definition (Complementary Sub-wffs)

Two sub-wffs F and G occurring in wffs S_1 and S_2 respectively are said to be *complementary* in them if either

(a) F and G are S-equivalent and occur with opposite polarities in S_1 and S_2, or

(b) F and $\sim G$ are S-equivalent and, F and G occur with same polarity in S_1 and S_2.

NC-resolution can be done on two sub-wffs only when they are complementary.

2.3 Unifiability of sub-wffs

The two complementary sub-wffs to be resolved upon have to be unified before resolving. We get the sequence of terms

in a sub-wff as described below.

2.3.1 Definition (Sequence of Terms)

The *Sequence of Terms* in a wff F can be obtained as below:

(a) If $F = P(t_1,...,t_n)$ where P is an n-place predicate symbol, then it is $(t_1,...,t_n)$.

(b) If $F = \sim F_1$, then it is the sequence of terms in F_1.

(c) If $F = (F_1 \; b \; F_2)$ for some binary connective b, then it is the sequence of terms in F_1 appended to the sequence of terms in F_2.

(d) If $F = (Qx) \; F_1$ where Q is some quantifier, and x is some variable in F_1, then it is the sequence of terms in F_1.

As an example, (x, y, z) is the sequence of terms in $P(x) \; V \; EAT(y, z)$. Two sub-wffs are *unifiable*, if the sequence of terms in one, and the sequence of terms in the S-equivalent form of the other are unifiable.

2.4 NC-resolution

NC-resolution is done on quantifier free wffs. The quantifiers are first moved to the left, and later *Skolemization* is performed to get the quantifier free form of the wffs.

Assuming that all separately quantified variables in a wff are relabeled apart, all the quantifiers can be moved to

the left by replacing every sub-wff in it, of the form

1. $[A \longleftrightarrow B]$ by $[(A \longrightarrow B) \& (B \longrightarrow A)]$
2. $[((\forall x)A) \longrightarrow B]$ by $(\exists x)[A \longrightarrow B]$
3. $[((\exists x)A \longrightarrow B]$ by $(\forall x)[A \longrightarrow B]$
4. $[A\ b\ (Qx)B]$ by $(Qx)[A\ b\ B]$, where b can be $\&$, V, or $\longrightarrow$
5. $\sim((\forall x)A)$ by $(\exists x)[\sim A]$
6. $\sim((\exists x)A)$ by $(\forall x)[\sim A]$

Now we shall see the formal definition of an NC-resolvent.

2.4.1 Definition (NC-resolvent)

For any quantifier free wffs S_1, S_2, F and G, if F occurs positively in S_1 $(S_1\langle F\rangle)$ and G occurs negatively in S_2 $(S_2\langle G\rangle)$, and θ is the *most general unifier* of F and G, such that $H = F\theta = G\theta$, then the result of simplifying

$$S_1\theta\{FALSE/H\} \vee S_2\theta\{TRUE/H\}$$

is the *NC-resolvent* of S_1, S_2. That is, H is replaced by *FALSE* in S_1 where it occurs positively, and by *TRUE* in S_2 where it occurs negatively, and the resulting wffs are disjoined and simplified.

If *FALSE* is reached as a resolvent, then the given set of wffs is unsatisfiable. NC-resolution is a sound and

complete proof procedure [Mur 82].

We shall consider one more example to illustrate the application of the inference rule.

Premises:

(1) The custom officials searched everyone who entered this country who was not a VIP.

(2) Some of the drug pushers entered this country and they were only searched by drug pushers.

(3) No drug pusher was a VIP.

Conclusion:

Some of the officials were drug pushers.

Let the predicate symbols be defined as below:

$E(x)$: *x entered the country,*

$V(x)$: *x was a VIP,*

$S(x, y)$: *y searched x,*

$C(x)$: *x was a custom official,*

$P(x)$: *x was a drug pusher.*

The FOL translation of the above problem, with the negated conclusion, consists of

S_1: $(\forall x)\ [(E(x)\ \&\ {\sim}V(x))\ \text{-->}\ (\exists y)\ (S(x, y)\ \&\ C(y))]$

S_2: $(\exists x)\ [P(x)\ \&\ E(x)\ \&\ (\forall y)\ (S(x, y)\ \text{-->}\ P(y))]$

S_3: $(\forall x)\ [P(x)\ \text{-->}\ {\sim}V(x)]$

S_4: $\sim(\exists x)\ [P(x)\ \&\ C(x)]$

The quantifier free version of the above set of wffs would be

S_1: $[(E(x)\ \&\ \sim V(x)) \dashrightarrow (S(x,\ f(x))\ \&\ C(f(x)))]$

S_2: $[P(a)\ \&\ E(a)\ \&\ (S(a,\ y) \dashrightarrow P(y))]$

S_3: $[P(x) \dashrightarrow \sim V(x)]$

S_4: $\sim[P(x)\ \&\ C(x)]$

The clausal version of this problem [ChL 73] consists of 7 clauses and its refutation takes 8 steps to give a null clause.

Non-clausal refutation of the above set requires only 6 steps and it is more readable.

$S_2 + S_3$ ---> R_1: $\sim V(a)$;$\{a/x\}$

{ derived that *a* is not a *VIP* }

$S_1 + R_1$ ---> R_2: $[E(a) \dashrightarrow (S(a,\ f(a))\ \&\ C(f(a)))]$;$\{a/x\}$

{ if *a* entered this country, *a* is searched by a custom official *f(a)* }

$S_2 + R_2$ ---> R_3: $[S(a,\ f(a))\ \&\ C(f(a))]$;$\{\}$

{ established that the custom official *f(a)* searched *a* }

$S_2 + R_3 \dashrightarrow R_4$: $[P(a)\ \&\ E(a)\ \&\ P(f(a))]$;$\{f(a)/y\}$

{derived that $f(a)$ is also a drug pusher}

$S_4 + R_4 \dashrightarrow R_5$: $\sim C(f(a))$;$\{f(a)/x\}$

{deduced that $f(a)$ is not a custom official}

But this contradicts the previously established assertion that $f(a)$ is a custom official. A resolution of R_5 with R_3 gives *FALSE*, thus proving the theorem.

$R_3 + R_5 \dashrightarrow R_6$: *FALSE* ;{}

Chapter 3

MANY-SORTED NON-CLAUSAL RESOLUTION

3.1 Introduction

Certain *axiomatic systems* involve more than one category of fundamental objects; like points, lines and planes in geometry. It is natural to use variables of different kinds with their ranges respectively restricted to categories of objects. We say

$$(\forall x, y: points), (\exists z: st\text{-}line)\ (L(x, z)\ \&\ L(y, z))$$

with an understanding that $L(x, y)$ means *"x lies on y"*, to state the axiom *"there is a straight line passing through any two points"*. The variables x and y are restricted to range over the category of points and z over st-lines. In other words, x and y are of the sort *point* and z of the sort *st-line*. An axiomatic theory so setup is called *Many-sorted*.

In the universe of all objects, a category is identified by its associated property. *is-a-point* is a property of objects of the sort *point*. In languages like First-Order Predicate Calculus, which provide variables of unres-

tricted range, we use such properties represented as unary predicates to reflect their sorts. The axiom is thus written as

$$(\forall x)(\forall y)\ [P(x)\ \&\ P(y) \dashrightarrow (\exists z)\ [St(z)\ \&\ L(x, z)\ \&\ L(y, z)]]$$

where $P(x)$ is read as *"x is a point"*, and $St(x)$ as *"x is a st-line"*. This is an unsorted version of the above many-sorted sentence. In the following text the prefix MS denotes many-sorted and US denotes unsorted.

Many-sorted logic has been studied in [Her 30], [Wan 52], [Hai 56] and others. The efforts of Herbrand [Her 30], and Schmidt [Sch 38] have been to prove that, something is provable in MS logic, iff its translation in US logic is provable.

Many-sorted logic provides many advantages. The resolution search space and the proof depth are either same or less compared to those in US. Many-sorted resolution uses constrained unification which prevents the generation of many useless resolvents.

A MS calculus for resolution and paramodulation has been described by Christoph Walther [Wal 82]. The resolution based on this calculus has been proved to be sound and complete. The non-clausal version of many-sorted resolution and a mechanism to convert the US set of sentences into an MS set were introduced by Jaya Raghavendra [Jay 87]. Many-

sorted Non-clausal Resolution has been proved to be sound and complete.

This chapter describes the methodology of converting the US set of sentences into an MS set and then performing the many-sorted non-clausal resolution on them.

3.2 Sorts: Formal Preliminaries

A *sort* is a category of elements in the universe of discourse. It is referred to, by a *sort-symbol*. T is a sort-symbol which refers to the entire universe of discourse. The following are the operators among sorts.

1. Sort union ($\bigcup$):

$s_1 \bigcup s_2$ is said to be the *sort union* of two sorts s_1 and s_2 and contains all elements (represented by constants, function symbols and variables) of the sort s_1 or s_2.

2. Sort intersection ($\bigcap$):

$s_1 \cap s_2$ is said to be the *sort intersection* of two sorts s_1 and s_2 and contains all elements of the sorts s_1 and s_2.

3. Subsort ($\subseteq$):

A sort s_1 is said to be a *subsort* of a sort s_2 ($s_1 \subseteq s_2$), if all elements of the sort s_1 are also of the sort s_2.

We have a lemma here for the properties of these operators.

Lemma 3.1

For any sort S (other than T),

(a) $S \bigcup T$ is the sort T

(b) $S \bigcap T$ is the sort S itself,

(c) $S \subseteq T$ is true, $T \subseteq S$ is false.

A sort relation of the form $S_1 \subseteq S_2$, where S_1 and S_2 are sorts, is called a *sort axiom*. $MAN \bigcup WOMAN \subseteq HUMAN$ is an example.

Given a set of sort axioms and two sorts S_1 and S_2, the repeated use of the following rules tells us if $S_1 \subseteq S_2$ is true under the set of sort axioms.

Let A, B and C be sorts.

1. $A \subseteq A$
2. *If* $A \subseteq B$ *and* $B \subseteq C$ *then* $A \subseteq C$
3. *If* $(A \bigcup B) \subseteq C$ *then* $A \subseteq C$ *and* $B \subseteq C$
4. *If* $A \subseteq (B \bigcap C)$ *then* $A \subseteq B$ *and* $A \subseteq C$
5. *If* $A \subseteq C$ *and* $B \subseteq C$ *then* $(A \bigcup B) \subseteq C$
6. *If* $A \subseteq B$ *and* $A \subseteq C$ *then* $A \subseteq (B \bigcap C)$

Also if by using the above rules on the set of sort axioms, it is found that there exists a set of sorts $S_1,\dots,S_n$, such that $S_1 \subseteq S_2,\dots,S_{n-1} \subseteq S_n, S_n \subseteq S_1$, then all S_i, $i=1,n$ can be replaced by S, where S is a new sort symbol in the set of sort axioms.

3.3 Many-sortedness

A wff is said to be a *many-sorted wff* if every term in it has a sort. The outermost symbol of a term is called the *terminal symbol,* and may be a variable, a constant or a function symbol. The sort of a term is the sort of its terminal symbol. *[t]* is used as an abbreviation for *'sort of t'.*

Consider a set of quantifier free wffs $U = \{U_1, U_2, ...\}$. We interpret U as a *conjunction* of its members. So any wff in U of the form A & B can be replaced by the wffs A and B . The generalized version of Davis and Putnam's rules [DaP 60] could be used to remove the redundant wffs from U . The one literal rule can remove all occurrences of the literal $P(t)$ and $\sim P(t)$, if either a wff $P(x)$ or $\sim P(x)$ is in U, x being a variable and t any term. We consider the translation of U, into its many-sorted version, in the next section.

3.3.1 Unsorted to Many-sorted

The many-sorted version of U would consist of

1. a set of sort axioms,

2. a set of many-sorted wffs M, and

3. sorts of all terms in M.

1 and 3 above form the *sort information* in U. This sort information is present in U_1 in the form of sub-wffs with

unary predicates, and should be removed to get M_i. A wff which gives the sort of a terminal symbol is called a *sort formula*, or *s.form* for short.

3.3.1.1 Definition (s.form)

A wff S is an *s.form* of a terminal symbol t if

(a) S is an atom with a unary predicate, and a term with t, or

(b) $S = P \& Q$ [$S = P \vee Q$] where P and Q are s.forms of t.

As an example, $[MAN(x) \vee WOMAN(x)] \& INTELLIGENT(x)$ is an s.form of x.

The *sort symbol*, in an s.form, can be obtained from the s.form by,

(a) replacing every atom by its predicate symbol, and

(b) replacing every $\vee$ by $\cup$, and every $\&$ by $\cap$.

The sort symbol in the above example is

$$[MAN \cup WOMAN] \cap INTELLIGENT.$$

All terms in U can be considered to be of the sort T, and for this reason the unsorted logic is also called the *one-sorted logic* [Wan 52]. Similarly, the terms in M can also be assigned the sort T, and updated as the sorting process continues.

If for a term t of the sort P, it is identified

that t also has a sort Q, then the sort of t will be

(a) P if $P \subseteq Q$

(b) Q if $Q \subseteq P$

(c) $P \cap Q$ otherwise.

The repeated use of the following rules, removes portions of wffs in U_i which represent sort information, giving the many-sorted wff M_i. We are augmenting the original set of 3 rules with an additional rule so that we can extract more sort information.

Rule 1:

Remove all wffs of the form $U' \dashrightarrow U''$ where U' and U'' are s.forms of a variable. If S' and S'' are the sort-symbols got from U' and U'' respectively, then $S' \subseteq S''$ is a sort axiom.

Rule 2:

If a wff U has a sub-wff F,

(a) which is positive in it, and

(b) has all occurrences of a variable v, and

(c) is of the form $F_1 \dashrightarrow F_2$, where F_1 is the s.form of v, and F_2 is some wff,

then F_1 can be removed, and the sort of v in U is given by the sort corresponding to F_1.

Rule 3:

If a wff U has a sub-wff F,

(a) which is positive in it, and

(b) has all occurrences of a constant or a function symbol c, and

(c) is of the form F_1 & F_2, where F_1 is an s.form of c, and F_2 is some wff or *TRUE*,

then F_1 can be removed, and the sort of c in U is given by the sort corresponding to F_1.

Rule 4:

If a wff U has a sub-wff F,

(a) which is positive in it, and

(b) has all occurrences of a constant or function symbol c, and

(c) is of the form $F_1 \dashrightarrow F_2$, where F_1 is the s.form of c and c has been found to be of the sort corresponding to F_1, and F_2 is some wff,

then F_1 can be removed.

As an example, consider the following set of unsorted sentences,

U_1: *MAN(x) --> HUMAN(x)*

U_2: *MAN(x) --> LIVES(x, Earth)*

U_3: *MAN(Ram) --> LIVES(Ram, Earth)*

U_4: *MAN(Ram)*

The MS version of the above example would be

Sort axioms: $MAN \subseteq HUMAN$

Sorts of terms: $x{:}MAN$, $Ram{:}MAN$

MS sentences: $LIVES(x, Earth)$

$LIVES(Ram, Earth)$.

Although every sort-symbol (other than T) has one or more unary predicate symbols, not all of them can be sort symbols. A unary predicate becomes a sort symbol in MS iff

(a) all occurrences of it are removed by the above rules, and

(b) there is at least one element of that sort.

The example which shows the necessity of such a restriction is given in [Jay 87].

The algorithm for converting an unsorted set of wffs U, to its many-sorted version M, is as below.

Step 1:

Apply rules 1, 2, 3 and 4, until no more application is possible. Properties of connectives, like $\sim(\sim A)$ is A, $A \rightarrow B$ is $\sim A \vee B$, etc. can be used to bring U_i into the form desirable for a rule application.

Step 2:

For every unary predicate which has an occurrence removed, ensure that all its occurrences are removed. If a unary predicate has some occurrences removed and some left in M, redo Step 1 without removing any occurrence of that predicate.

Step 3:

For all terms in M, check the existence of an element of the sort of the term. If for some term t, $[t]$ is S, and the existence of an element of the sort S cannot be determined, then S cannot be a sort in MS. Reapply the algorithm ensuring that no occurrence of, (a) S if S is a predicate symbol, or (b) either S_1 or S_2, if S is $S_1 \cap S_2$, is removed.

3.4 Resolution and Unification

The many-sorted non-clausal resolution (MNC-resolution for short) is the NC-resolution of [Mur 82] with MS unification.

3.4.1 Definition (MNC-resolvent)

If P is a sub-wff that occurs positively in the wff F_1 and Q occurs negatively in F_2, and θ is the MS mgu of P and Q, then the result of simplifying

$$F_1\theta\{FALSE/P\theta\} \vee F_2\theta\{TRUE/Q\theta\}$$

is the *MNC-resolvent* of F_1 and F_2.

In MS every term has a sort. The MS unification would be the usual unification process of finding an mgu, and then checking the sort compatibility of every substitution in the mgu.

A substitution of a term t_1 in place of a term t_2, written as $\{t_1/t_2\}$ is *sort compatible* if $[t_1] \subseteq [t_2]$.

So in MNC-resolution only sort compatible substitutions are done.

3.4.2 Weakening rule

While constraining unification in MS is necessary, it is not enough to make MNC-resolution complete without this additional rule of inference. The following example [Wal 82] illustrates this point.

U_1: $D(x) \;\text{-->}\; B(x)$

U_2: $B(x) \;\text{-->}\; A(x)$

U_3: $D(x) \;\text{-->}\; C(x)$

U_4: $C(x) \;\text{-->}\; A(x)$

U_5: $D(D_1)$

U_6: $B(u) \;\text{-->}\; P(u)$

U_7: $C(v) \;\text{-->}\; {\sim}P(v)$

Although the above set of wffs is unsatisfiable, the MS version of it given below, cannot be proved unsatisfiable by MNC-resolutions.

Sort axioms: $D \subsetneq B$, $B \subsetneq A$

$D \subsetneq C$, $C \subsetneq A$

Sorts of terms: $D_1 : D$, $u : B$, $v : C$

MS sentences:

M_1: $P(u)$

M_2: ${\sim}P(v)$

As a solution to this problem Walther [Wal 82] has pro-

posed an additional rule of inference called the *Weakening rule.*

Weakening rule:

> If U is a wff with a variable v in it, and t is a term such that $[t] \subseteq [v]$, then the wff U with t substituted for v, (i.e., $U\theta$, $\theta = \{t/v\}$) is called the *weakened variant* of U.

Using the above rule we can get the weakened variant of M_1 as

$$M_3: P(D_1) \quad ; \{D_1/u\}$$

Now, resolving M_3 and M_2 with the substitution $\{D_1/v\}$ gives $FALSE$, thus proving the unsatisfiability of the set.

We shall reconsider the example (*drug pusher - custom official problem*) given in Chapter 2 to illustrate the application of many-sorted rules of inference.

The quantifier free set of the wffs is

S_1: $[(E(x) \,\&\, \sim V(x)) \dashrightarrow (S(x, f(x)) \,\&\, C(f(x)))]$

S_2: $[P(a) \,\&\, E(a) \,\&\, (S(a,y) \dashrightarrow P(y))]$

S_3: $[P(z) \dashrightarrow \sim V(z)]$

S_4: $\sim[P(u) \,\&\, C(u)]$

The many-sorted version of the above set would be

Sort axioms: None.

Sorts of terms: $x:E,\quad f(x):C,\quad a:E,$

$y:T,\quad z:T,\quad u:C.$

MS sentences:

M_1: $[\sim V(x) \dashrightarrow S(x, f(x)]$

M_2: $[P(a)\ \&\ (S(a, y) \dashrightarrow P(y))]$

M_3: $[P(z) \dashrightarrow \sim V(z)]$

M_4: $\sim P(u)$.

The otherwise possible resolution of M_4 with M_2 on $\sim P(u)$ and $P(a)$ respectively is now constrained because of the sort incompatibility of the terms u and a.

As opposed to the unsorted NC-refutation of 6 steps, many-sorted NC-refutation as given below takes only 4 steps.

$M_2 + M_3 \dashrightarrow R_1$: $\sim V(a)$;$\{a/z\}$

{ a of category E is not a VIP }

$M_1 + R_1 \dashrightarrow R_2$: $S(a, f(a))$;$\{a/x\}$

{ some $f(a)$ of category C searched a }

$M_2 + R_2 \dashrightarrow R_3$: $[P(a)\ \&\ P(f(a))]$;$\{f(a)/y\}$

{ established that $f(a)$ is also a drug pusher }

$M_4 + R_3 \dashrightarrow R_4$: FALSE ;$\{f(a)/u\}$

Thus, the proof depth gets reduced in many-sorted resolution.

Chapter 4

EQUALITY BASED NON-CLAUSAL RESOLUTION

The axioms of equality play a crucial role in automated reasoning because of their almost universal application in particular theories. The axioms of equality are,

E: 1. $x = x$ *reflexivity,*

2. $x = y \longrightarrow y = x$ *symmetry,*

3. $x = y \,\&\, y = z \longrightarrow x = z$ *transitivity,*

4. $x_i = x_o \,\&\, P(x_1,\ldots,x_i,\ldots,x_n) \longrightarrow P(x_1,\ldots,x_o,\ldots,x_n)$

   substitution in predicates for each n-place predicate symbol,

5. $x_i = x_o \longrightarrow f(x_1,\ldots,x_i,\ldots,x_n) = f(x_1,\ldots,x_o,\ldots,x_n)$

   substitution in functions for each n-place function symbol.

A set U of wffs is said to be *E-unsatisfiable,* if $U \cup E_U$ is unsatisfiable, where E_U denotes the axioms of equality associated with U. The number of equality axioms depends on the number of distinct predicate and function symbols in U and the number of arguments each such symbol has. Thus the set E_U can be very large and often is larger

than *U* itself. So explicit use of the axioms of equality in a refutation proof can prove very expensive. Also it can make certain strategies inapplicable [DiH 86], [MaW 86].

In response to these problems, some theorem-proving researchers, such as Kowalski [Kow 79] have paraphrased their theories to avoid explicit use of the equality axioms. Others have used special inference rules; *paramodulation* (introduced by Wos and Robinson[WoR 69]) and *E-resolution* (introduced by Morris[Mor 69]) have been found to be effective. Variations of these rules, proposed by Boyer and Moore[BoM 79] and Digricoli[DiH 86] are being widely used.

Even though paramodulation and E-resolution rules can dispense with the equality axioms, some problems still remain, such as no general purpose algorithm to control paramodulation and incompleteness due to the uniform use of the innermost inequalities in E-resolution [DiH 86]. To overcome these problems, Digricoli[DiH 86] introduced an improved version of E-resolution, namely *Resolution by Unification and Equality (RUE)* which facilitates the definition of heuristics that lead to more efficient searches for refutations.

By a single application of either of these rules, we can derive conclusions that would require several steps if equality axioms were explicitly used. The proofs are markedly shorter, and the search spaces are even more dramatically compressed because the axioms and intermediate steps are not required. Theorem- proving systems using

certain heuristics [DiH 86] developed in the context of RUE resolution produce very brief and incisive proofs.

4.1 Introduction

The non-clausal analogues of Paramodulation and E-resolution have been introduced by Manna & Waldinger [MaW 86] where the rules can be applied to free form sentences with a full set of logical connectives.

Digricoli has introduced RUE resolution to overcome the deficiencies in Paramodulation and E-resolution for sentences in clausal form.

The most efficient clausal refutation procedure incorporating equality seems to be RUE. However, the non-clausal analogue using RUE has not been done yet and this is the motivation for the work described in this chapter.

The basic definitions, both in *open form* and *strong form* (the terms *open form* and *strong form* are defined shortly), for the non-clausal analogue of RUE resolution which we call *Non-Clausal Resolution by Unification and Equality (NCRUE)* are given in this chapter. We extend the notions of disagreement sets, viability, RUE unifier, and the equality restriction introduced by Digricoli[DiH 86] for RUE resolution to NCRUE resolution. The reader is assumed to be familiar with the notations used in [Rob 65], [DiH 86], [Mur 82] and [MaW 86]. We prove the soundness and completeness of NCRUE resolution in open form, completeness for viability criterion, completeness for the equality restriction

and finally give a conjecture for the completeness of NCRUE resolution in strong form. The application of the inference rule is illustrated by examples.

4.2 NCRUE Resolution

In NCRUE resolution, we can resolve two wffs even if they fail to unify by introducing certain *conditions* into the resolvent. We first define disagreement sets and then describe the two rules of inference. These rules are in *open form* since we are free to choose both the substitution and disagreement set to be used.

4.2.1 A Disagreement Set of a Pair of Terms

If s, t are nonidentical terms the singleton set $\{s:t\}$, is the *origin disagreement set.* If s, t have the form $f(a_1,...,a_k)$, $f(b_1,...,b_k)$, then the set of pairs of corresponding arguments that are not identical is the *top-most disagreement set.* Furthermore, if D is a disagreement set, then D' formed by replacing any member of D by the elements of its disagreement set, is also a disagreement set. If s, t are identical terms, the empty set is the only disagreement set.

For example, the pair of terms, $f[a, h(b, g(c))]:f[b, h(c, g(d))]$, has the disagreement sets

D_1: $\{f[a, h(b, g(c))]:f[b, h(c, g(d))]\}$

the origin disagreement set,

and finally give a conjecture for the completeness of NCRUE resolution in strong form. The application of the inference rule is illustrated by examples.

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If s, t are nonidentical terms the singleton set $\{s:t\}$, is the *origin disagreement set.* If s, t have the form $f(a_1,...,a_k)$, $f(b_1,...,b_k)$, then the set of pairs of corresponding arguments that are not identical is the *top-most disagreement set.* Furthermore, if D is a disagreement set, then D' formed by replacing any member of D by the elements of its disagreement set, is also a disagreement set. If s, t are identical terms, the empty set is the only disagreement set.

For example, the pair of terms, $f[a, h(b, g(c))]:f[b, h(c, g(d))]$, has the disagreement sets

D_1: $\{f[a, h(b, g(c))]:f[b, h(c, g(d))]\}$

the origin disagreement set,

D_2: $\{a:b,\ h(b,\ g(c)):h(c,\ g(d))\}$

the topmost disagreement set of two pairs,

D_3: $\{a:b,\ b:c,\ g(c):g(d)\}$,

D_4: $\{a:b,\ b:c,\ c:d\}$.

4.2.2 A Disagreement Set of Complementary Literals

A disagreement set of complementary literals, $P(s_1,...,s_n)$, $\sim P(t_1,...,t_n)$, is defined as the union

$$D = \bigcup_{i=1}^{n} D_i,$$

where D_i is a disagreement set of the corresponding arguments s_i, t_i.

The topmost disagreement set of $P(s_1,...,s_n)$, $\sim P(t_1,...,t_n)$ is the set of pairs of corresponding arguments that are not identical.

4.2.3 A Disagreement Set of a Set of Literals

A disagreement set of m positive(negative) literals, $L_1,...,L_m$, is defined as the union

$$D = \bigcup_{i=1}^{m-1} D_i,$$

where D_i is a disagreement set of a pair of literals, L_i, L_{i+1}.

For example, a disagreement set of 3 literals, $P(a,\ b)$, $P(c,\ d)$, $P(e,\ f)$, would be

$$D: \{a:c,\ b:d,\ a:e,\ b:f\}.$$

D is the topmost disagreement set of these m literals, when every D_i is the topmost disagreement set.

In the following text, by *"unifying to the maximum extent possible"* we mean that we use the maximum *most general partial unifier (mgpu)*. For example, when we unify the terms $f(x, g(y, a)), f(b, g(c, d))$ we use the substitution $\{b/x, c/y\}$ instead of any of the other partial unifiers $\{b/x\}, \{c/y\}$.

4.2.4 The NCRUE rule of inference (open form)

Let a wff U_i contain atoms $L_{i1},\ldots,L_{in}$, such that θ_i is a substitution that unifies $\{L_{i1},\ldots,L_{in}\}$ to the maximum extent possible, and D_i is a conjunction of equalities to be satisfied as specified by a disagreement set of $\{L_{i1},\ldots,L_{in}\}\theta_i = \{L'_{i1},\ldots,L'_{ik}\}$; $1 \leq k \leq n$. Under the condition that D_i is true, $\{L'_{i1},\ldots,L'_{ik}\} = \{L_i\}$.

Let another wff U_j contain atoms $L_{j1},\ldots,L_{jm}$, such that θ_j is a substitution that unifies $\{L_{j1},\ldots,L_{jm}\}$ to the maximum extent possible, and D_j is a conjunction of equalities to be satisfied as specified by a disagreement set of $\{L_{j1},\ldots,L_{jm}\}\theta_j = \{L'_{j1},\ldots,L'_{jl}\}$; $1 \leq l \leq m$. If D_j is true, then $\{L'_{j1},\ldots,L'_{jl}\} = \{L_j\}$.

Let θ be a substitution that unifies $\{L_i, L_j\}$ and D be a conjunction of equalities to be satisfied as specified by a disagreement set of $\{L_i, L_j\}\theta = \{L'_i, L'_j\}$. If D

If L occurs positively in $U_i(\theta_i \bullet \theta) = U_i'$ and negatively in $U_j(\theta_j \bullet \theta) = U_j'$, then the *NCRUE resolvent* of U_i and U_j is

$$(D_i \ \& \ D_j \ \& \ D) \ \text{-->} \ (U_i\{FALSE/L\} \lor U_j\{TRUE/L\}).$$

If L is negative in U_i' and positive in U_j', then we have the following dual NCRUE resolvent:

$$(D_i \ \& \ D_j \ \& \ D) \ \text{-->} \ (U_i\{TRUE/L\} \lor U_j\{FALSE/L\}).$$

We may use the rule for symmetry, defined in section 4.2.6, with NCRUE applied to equality literals.

4.2.5 The NCNRF rule of inference (open form)

Let a wff U_i contain equality atoms $=_1,...,=_n$, such that θ_i is a substitution that unifies $\{=_1,...,=_n\}$ to the maximum extent possible, and D_i is a conjunction of equalities to be satisfied as specified by a disagreement set of $\{=_1,...,=_n\}\theta_i = \{=_1',...,=_k'\}$; $1 \leq k \leq n$. If D_i is true, then $\{=_1',...,=_k'\} = \{=\}$.

If $=$ has the form $t_1 = t_2$ and occurs negatively in $U_i\theta_i$, and θ is a substitution and D is a conjunction of equalities to be satisfied as specified by a disagreement set of $t_1\theta$, $t_2\theta$, then the *Non-Clausal Negative Reflective Function (NCNRF) resolvent* of U_i is

$$(D_i \ \& \ D) \longrightarrow U_i(\theta_i \circ \theta)\{TRUE/t_1\theta = t_2\theta\}.$$

This is called the *negative reflective function rule* because its most important application is when $t_1 = t_2$ has the form $f(a_1,\ldots,a_k) = f(b_1,\ldots,b_k)$ and occurs negatively in the wff.

4.2.6 Rules for Symmetry

Our two rules of inference as defined contain the implicit application of all the axioms of equality except symmetry. For the latter, we must apply the following rules:

(1) If a wff U_i contains equality atoms $a_1 = a_2$ and $b_1 = b_2$, and when we perform the *merging (factoring)* on these two literals, factor them to two different conditions D_{i1} and D_{i2}, where D_{i2} is obtained by interchanging the arguments of equality in one of the factored literals.

We form

$$D_{i1}: a_1 = b_1 \ \& \ a_2 = b_2,$$
$$D_{i2}: a_2 = b_1 \ \& \ a_1 = b_2.$$

(2) When we resolve U_i with another wff U_j containing negative equality literal $c_1 =^- c_2$, for each D_i above, form two resolvents, the second obtained by interchanging the arguments of equality in one of the input

equality literals. Thus, we form with D_{i1}

$$U_i\langle a_1 = a_2, b_1 = b_2\rangle$$

$$\text{------}\ U_j\langle c_1 =^{-} c_2\rangle$$

$$(D_{i1} \ \& \ D_1) \longrightarrow (U_i\{FALSE/a_1 = a_2,\ FALSE/b_1 = b_2\} \vee U_j\{TRUE/c_1 =^{-} c_2\})$$

where

$$D_1 : a_1 = c_1 \ \& \ a_2 = c_2$$

and also the second resolvent

$$U_i\langle a_2 = a_1, b_1 = b_2\rangle$$

$$\text{------}\ U_j\langle c_1 =^{-} c_2\rangle$$

$$(D_{i1} \ \& \ D_2) \longrightarrow (U_i\{FALSE/a_2 = a_1,\ FALSE/b_1 = b_2\} \vee U_j\{TRUE/c_1 =^{-} c_2\})$$

where

$$D_2 : a_2 = c_1 \ \& \ a_1 = c_2$$

Similarly, we form two resolvents corresponding to D_{i2}, when we use the second factor of U_i.

The above is sufficient to build in the entire effect of symmetry. The virtue of using this rule, in place of including the axiom of symmetry in U, is that it leads to a less redundant application of this axiom while preserving completeness. Furthermore, it eliminates the need to store the symmetry variants of a wff. For example, we store one copy of $(D \longrightarrow U_i)$, where D contains n equalities, instead of 2^n symmetry variants originating in permuting the order of arguments in the equalities of D.

4.2.7 NCRUE-NCNRF Deduction

Given a set of wffs U, an *NCRUE-NCNRF deduction* of from U is a finite sequence of $R_1,\ldots,R_n$ such that

each R_i is either a wff of U or an NCRUE-NCNRF resolvent of wffs preceding R_i and where R_n is R.

4.2.8 NCRUE Refutation of *U*

A deduction of *FALSE* from *U* is called an *NCRUE Refutation of U.*

We consider a small problem from [Pel 86] to illustrate the rules of inference.

The set of wffs consists of just two wffs:

$$U_1: f(x, f(y, z)) = f(f(x, y), z)$$

$$U_2: f(a, f(b, f(c, d))) \neq f(f(f(a, b), c), d)$$

Here we can resolve these two wffs with the mgpu $\{a/x, d/z\}$ and the disagreement set $\{f(b, f(c, d)) : f(y, d), f(f(a, b), c) : f(a, y)\}$ to give NCRUE resolvent

$$U_1 + U_2 \longrightarrow R_1:$$

$$[f(b, f(c, d)) = f(y, d) \ \& \ f(f(a, b), c) = f(a, y)]$$

$$\longrightarrow FALSE \vee \sim TRUE$$

which simplifies to

$$\sim[f(b, f(c, d)) = f(y, d) \ \& \ f(f(a, b), c) = f(a, y)].$$

Since we are resolving on two equality literals, we

will apply the rules for symmetry as well. If we interchange the arguments in the equality of U_1 then we have the following resolvent with the substitution $\{f(f(a, b), c)/x, f(b, f(c, d))/z\}$ and the disagreement set $\{f(f(f(a, b), c), y):a, f(y, f(b, f(c, d))):d\}$

$$\sim[f(f(f(a, b), c), y) = a \ \& \ f(y, f(b, f(c, d))) = d].$$

For simplicity, we do not give the resolvents generated from the application of the rules for symmetry in further resolutions unless it is required.

Now R_1 can be resolved on the sub-wff $f(b, f(c, d)) = f(y, d)$ with U_1 (variables x, y, z renamed to x_1, y_1, z_1) with the substitution $\{b/x_1, c/y_1, d/z_1\}$ and the disagreement set $\{f(y, d):f(f(b, c), d)\}$ to give

$U_1 + R_1 \text{ ---> } R_2$:

$$f(y, d) = f(f(b, c), d) \text{ --> } \sim(f(f(a, b), c) = f(a, y)).$$

Here we could have used the additional substitution $\{f(b, c)/y\}$ so that we get R_2 as $\sim(f(f(a, b), c) = f(a, f(b, c))$. But to illustrate how NCNRF rule of inference works, we have omitted this.

The equality $f(y, d) = f(f(b, c), d)$ is negative in R_2 and with the substitution $\{f(b, c)/y\}$ and null disagreement set we can apply NCNRF rule to give

$R_2 \longrightarrow_{NCNRF} R_3$: $TRUE \longrightarrow \sim(f(f(a, b), c) = f(a, f(b, c)))$

which simplifies to

$$\sim(f(f(a, b), c) = f(a, f(b, c))).$$

Now, with the rule for symmetry applied to U_1, we can resolve R_3 with U_1 with the substitution $\{a/x, b/y, c/z\}$ and null disagreement set to give $FALSE$, thus refuting the given set.

$U_1 + R_3 \longrightarrow R_4$: $FALSE$.

We shall now consider an example from *Group Theory* to illustrate the rules of inference.

Group Theory axioms:

U_1: $f(e, x) = x$	*Left identity*
U_2: $f(x, e) = x$	*Right identity*
U_3: $f(g(x), x) = e$	*Left inverse*
U_4: $f(x, g(x)) = e$	*Right inverse*
U_5: $f(f(x, y), z) = f(x, f(y, z))$	*Associativity*

Theorem:

The inverse of the inverse of x is x.

i.e. $(\forall x)\ g(g(x)) = x$.

The negated theorem, when skolemized, is

U_6: $\sim(g(g(a)) = a)$.

The NCRUE-NCNRF refutation of this theorem is:

$U_2 + U_6 \longrightarrow R_1$: $\sim(f(g(g(a)), e) = a)$;$\{g(g(a))/x\}$

$U_5 + R_1 \longrightarrow R_2$:

$\sim[f(f(g(g(a)), y), z) = a\ \&\ f(y, z) = e]$;$\{g(g(a))/x\}$

$U_1 + R_2 \longrightarrow R_3$:

$e = f(g(g(a)), y) \longrightarrow \sim(f(y, a) = e)$;$\{a/x, a/z\}$

$U_3 + R_3 \longrightarrow R_4$: $\sim(f(g(a), a) = e)$;$\{g(a)/x, g(a)/y\}$

$U_3 + R_4 \longrightarrow R_5$: FALSE ;$\{a/x\}$

Some more interesting examples are given in Chapter 10.

4.3 Soundness and Completeness of NCRUE resolution

We prove here the soundness and completeness of NCRUE resolution in open form, and in later sections, we give the results for strong form.

Theorem 4.1

The NCRUE rule of inference is sound.

Proof:

The proof is derived from the substitution axioms for predicates and from the soundness of NCR [Mur 82].

We must prove

$$U_i\langle P^+(s_1,\ldots,s_n)\rangle\theta \;\&\; U_j\langle P^-(t_1,\ldots,t_n)\rangle\theta \longrightarrow$$
$$[D \longrightarrow (U_i\theta\{FALSE/P(s_1,\ldots,s_n)\theta\} \vee U_j\theta\{TRUE/P(t_1,\ldots,t_n)\theta\})]$$

where D is a conjunction of equalities to be satisfied as specified by a disagreement set of $P(s_1,\ldots,s_n)\theta$, $P(t_1,\ldots,t_n)\theta$ and where $P(s_1,\ldots,s_n)$ is positive in U_i and $P(t_1,\ldots,t_n)$ is negative in U_j as indicated by P^+ and P^- respectively.

Suppose the antecedent is true in some model M. If D is false in M, then the consequent is true in M. If D is true in M, then $s_i\theta = t_i\theta$ in M for $i=1, \mid D \mid$, and thus by the substitution axiom for predicates, $P(s_1,\ldots,s_n)\theta$, $P(t_1,\ldots,t_n)\theta$ have the same truth value in M. Now this is the same as resolving U_i and U_j non-clausally with the substitution θ, and by the soundness of NCR, the consequent is true in M.

Thus, in all models in which the antecedent is true, the consequent must be true. □

Theorem 4.2

The NCNRF rule of inference is sound.

Proof:

When a wff U_i contains an equality, $t_1 = t_2$, of the form $f(a_1,\ldots,a_k) = f(b_1,\ldots,b_k)$ which occurs negatively

substitution axiom

$$x_o = x_k \dashrightarrow f(\ldots,x_{k-1},x_k,x_{k+1},\ldots) = f(\ldots,x_{k-1},x_o,x_{k+1},\ldots)$$

which states that a function does not change its value if one (or more) of its arguments is replaced by an equal argument.

We must prove

$$U_i\langle f(a_1,\ldots,a_k) =^- f(b_1,\ldots,b_k)\rangle\theta \;{-}{-}{-}{>}$$
$$[D \dashrightarrow U_i\theta\{TRUE/f(a_1,\ldots,a_k)\theta =^- f(b_1,\ldots,b_k)\theta\}]$$

where D is a conjunction of equalities to be satisfied as specified by a disagreement set of $f(a_1,\ldots,a_k)\theta$, $f(b_1,\ldots,b_k)\theta$.

Suppose the above antecedent is true in some model M. If D is false in M, the consequent is true in M. If D is true in M, then $a_i\theta = b_i\theta$, $i=1,k$, in M, and hence $f(a_1,\ldots,a_k)\theta = f(b_1,\ldots,b_k)\theta$ is true in M, due to the above equality substitution axiom. Since the antecedent is true in M, it follows that $U_i\theta\{TRUE/f(a_1,\ldots a_k)\theta =^- f(b_1,\ldots,b_k)\theta\}$ is true in M. Thus, for all M, if the antecedent is true, the consequent must be true.

When t_1, t_2 differ but do not represent the same function, then the only disagreement set is the origin disagreement $\{t_1:t_2\}$ and the application of NCNRF to nega-

tive $t_1 = t_2$ is redundant. We also note that $\sim(x = x)$ $\longrightarrow$ *FALSE* by NCNRF, which, in effect, applies the reflexive axiom. □

Theorem 4.3 Completeness of NCRUE Resolution (Open form)

If U is an E-unsatisfiable set of wffs, there exists an NCRUE-NCNRF deduction of *FALSE* from U.

This result is for resolution in open form since we have not specified the substitution or disagreement set to be used in applying the inference rules.

The proof of completeness is derived from Horn sets and ground unit wffs. A set of clauses is in *Horn form* if each of its clauses has at most one positive literal. In proving completeness, we apply the following theorem from unification resolution.

Theorem 4.4

If S is an unsatisfiable Horn set, there exists a factor-free, positive unit refutation of S.

Proof: Refer to [DiH 86].

Theorem 4.5 RUE Completeness for Horn Sets

If S is an E-unsatisfiable Horn set, there exists a ground RUE-NRF deduction of the empty clause from S.

Proof: Refer to [DiH 86].

Theorem 4.6 NCRUE Completeness for Ground Unit Wffs

If U is an E-unsatisfiable set of ground unit wffs,

there exists a ground NCRUE-NCNRF deduction of *FALSE* from *U*.

Proof:

The proof is derived from the *RUE Completeness for Horn sets* and the *fact* that any equality axiom when converted to clause form gives only one clause.

A set of unit wffs is a set of unit clauses too. Each equality axiom whether in clause form or in well formed form contains at most one positive literal and hence is Horn.

Since U is E-unsatisfiable, $U \cup E$ is unsatisfiable. Since E is a Horn set and U is also a Horn set, $U \cup E$ is Horn. By Theorem 4.5, there exists a ground RUE-NRF deduction of empty clause from U, an E-unsatisfiable Horn set.

A binary resolution may be viewed as replacing matched positive and negative atoms by *FALSE* and *TRUE*, and forming the disjunction of the resultant clauses. Then the resolvent would contain (possibly multiple occurrences of) the literals *FALSE* and ~*TRUE*, which can be dropped by truth-functional simplification. The idea behind NCR is also a similar replacement of matching atoms in parent wffs by truth values. The inferred formula is then the disjunction of the resultant formulae, simplified (or reduced) truth-functionally. Because in NCR wffs are involved which are not necessarily clauses the reduction may involve more

than just dropping the atoms resolved upon.

But in the present case both clausal and non-clausal refutations will have the same resolution sequence. Hence, a positive unit refutation in clausal version can be rewritten in non-clausal form without changing the sequence.

By Theorem 4.4, $U \cup E$ has a positive unit refutation. We show how to rewrite R, the unit refutation of $U \cup E$ by unification resolution, as a refutation without the equality axioms using the NCRUE-NCNRF rules of inference.

With respect to U, the following are the equality axioms in well-formed form:

E: 1. $x = x$ *reflexivity,*

2. $x = y \longrightarrow y = x$ *symmetry,*

3. $x = y \;\&\; y = z \longrightarrow x = z$ *transitivity,*

4. $x_i = x_o \;\&\; P(x_1,\dots,x_i,\dots,x_n) \longrightarrow P(x_1,\dots,x_o,\dots,x_n)$

substitution in predicates for each n-place predicate symbol,

5. $x_i = x_o \longrightarrow f(x_1,\dots,x_i,\dots x_n) = f(x_1,\dots,x_o,\dots,x_n)$

substitution in functions for each n-place function symbol.

Let R be a ground, positive unit refutation of $U \cup E$ by unification resolution, which is factor free. We now show how to remove each of the equality axioms from R

(3) In R every use of transitivity axiom will appear in the form

$$t_1 = t_2 \;\&\; t_2 = t_3 \;\text{-->}\; t_1 = t_3$$
$$|\text{-----}\quad t_1 = t_2$$
$$\sim(t_2 = t_3) \;V\; (t_1 = t_3)$$
$$|\text{-----}\quad t_2 = t_3$$
$$t_1 = t_3$$
$$|\text{-----}\quad U_i\langle t_1 =^- t_3\rangle$$
$$U_i\{TRUE/t_1 =^- t_3\}$$

Since R is a positive unit refutation, every use of the transitivity axiom in R must be in the above form (or possibly steps (1) and (2) interchanged). We replace the above sequence by

$$U_i\langle t_1 =^- t_3\rangle \qquad\qquad \text{(NCRUE)}$$
$$|\text{-----}\quad t_1 = t_2$$
$$t_2 = t_3 \;\text{-->}\; U_i\{TRUE/t_1 =^- t_3\}$$
$$|\text{-----}\quad t_2 = t_3$$
$$U_i\{TRUE/t_1 =^- t_3\}$$

(4) In R every use of the substitution axiom for predicates will appear in the form

$$(t_o = t_k \;\&\; P(\ldots,t_k,\ldots)) \;\text{-->}\; P(\ldots,t_o,\ldots)$$
$$|\text{-----}\quad t_o = t_k$$
$$P(\ldots,t_k,\ldots) \;\text{-->}\; P(\ldots,t_o,\ldots)$$
$$|\text{-----}\quad P(\ldots,t_k,\ldots)$$
$$P(\ldots,t_o,\ldots)$$
$$|\text{-----}\quad U_i\langle P^-(\ldots,t_o,\ldots)\rangle$$
$$U_i\{TRUE/P^-(\ldots,t_o,\ldots)\}$$

with possibly the order of steps (1) and (2) interchanged. In the above, the corresponding arguments of P^+, P^- are identical except at t_o, t_k. We replace the above by

$$
\begin{array}{ll}
U_i\langle P^-(\ldots,t_o,\ldots)\rangle & \text{(NCRUE)} \\
\quad | \text{-----} \; P(\ldots,t_k,\ldots) & \\
t_o = t_k \dashrightarrow U_i\{TRUE/P^-(\ldots,t_o,\ldots)\} & \\
\quad | \text{-----} \; t_o = t_k & \\
U_i\{TRUE/P^-(\ldots,t_o,\ldots)\} &
\end{array}
$$

(5) In R every use of the substitution axiom for functions will appear in the form

$$
\begin{array}{l}
t_o = t_k \dashrightarrow f(\ldots,t_o,\ldots) = f(\ldots,t_k,\ldots) \\
\quad | \text{-----} \; t_o = t_k \\
f(\ldots,t_o,\ldots) = f(\ldots,t_k,\ldots) \\
\quad | \text{-----} \; U_i\langle f(\ldots,t_o,\ldots) =^- f(\ldots,t_k,\ldots)\rangle \\
U_i\{TRUE/f(\ldots,t_o,\ldots) =^- f(\ldots,t_k,\ldots)\}
\end{array}
$$

where the corresponding arguments of $f(\ldots,t_o,\ldots)$ and $f(\ldots,t_k,\ldots)$ are identical, except at t_k. We replace the above by

$$
\begin{array}{l}
U_i\langle f(\ldots,t_o,\ldots) =^- f(\ldots,t_k,\ldots)\rangle \\
\quad | \text{-----} \; \text{NCNRF} \\
t_o = t_k \dashrightarrow U_i\{TRUE/f(\ldots,t_o,\ldots) =^- f(\ldots,t_k,\ldots)\} \\
\quad | \text{-----} \; t_o = t_k \\
U_i\{TRUE/f(\ldots,t_o,\ldots) =^- f(\ldots,t_k,\ldots)\}
\end{array}
$$

By applying the above transformations (1) to (5) we can transform R a positive unit refutation of $U \cup E$ using

unification resolution, into a positive unit NCRUE-NCNRF refutation of U. Hence if U is an E-unsatisfiable ground set of unit wffs, there exists a ground NCRUE-NCNRF refutation of U, which is a factor-free, positive unit refutation. □

Theorem 4.7 Completeness of NCRUE Resolution (Open form, Ground case)

If U is an E-unsatisfiable set of ground wffs, there exists an NCRUE-NCNRF deduction of *FALSE* from U.

Proof:

Let $k(U)$ be the total number of appearances of literals in U minus the number of wffs in U. Our proof is by induction on $k(U)$, *the excess literal parameter.*

For example, $k(U)$ for the following set of wffs U,

$$U_1: (P \longleftrightarrow Q) \vee R$$

$$U_2: \sim P \,\&\, Q \,\&\, \sim R$$

is 4.

If $k(U) = 0$, then U consists of only ground unit wffs. By Theorem 4.6, there is a unit NCRUE refutation of U that is factor free.

Now suppose our theorem holds for U ground and $k(U) \leq n$. We show that it holds for $k(U) = n + 1$.

Suppose ground U has $n + 1$ excess literals. Consider

a wff U_1 of U having an excess literal L. Depending on how L is *effectively* connected to other sub-wff, say F_1, in U_1, we will have 4 different cases.

Case(i): $F_1 \vee L$

Define U' as $U - \{U_1\}$ and U_1' as $U_1 - \{L\}$, and U_1'' as $U_1 - \{F_1\}$. Since U is E-unsatisfiable, so are $U' \cup \{U_1'\}$ and $U' \cup \{U_1''\}$. Since both of these have n or fewer excess literals, they both have NCRUE-NCNRF refutations.

Now consider the refutation R_1 of $U' \cup \{U_1'\}$. Modify R_1 by adding back the literal L to the wff U_1'. This may give an NCRUE-NCNRF refutation of U in which case we derive a *FALSE* without the participation of L. Otherwise, this will give an NCRUE derivation of L from U. By appending to this derivation the portion of the refutation of $U' \cup \{U_1''\}$ where L is refuted, we get an NCRUE-NCNRF refutation of ground U, which has $n + 1$ excess literals.

We note that if $U_1' \cup \{L\}$ is used m times as an input wff in R_1, then the residue in R_1 is L taken m times, $L \vee L \vee \ldots \vee L$, which is merged to the unit wff L. Hence, the refutation we have constructed to refute U is not necessarily factor free.

Case(ii): F_1 & L

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{L\}$ and U_1'' as $U_1 - \{F_1\}$. Since U is E-unsatisfiable, either of $U' \cup \{U_1'\}$ or $U' \cup \{U_1''\}$ is E-unsatisfiable or both --

at least $U' \cup \{U_1'\}$ will give a derivation of $\sim L$.

Let us assume that $U' \cup \{U_1'\}$ is E-unsatisfiable. Since it has n or fewer excess literals, there is an NCRUE-NCNRF refutation. This will give us an NCRUE-NCNRF refutation of U as well, because when we reduce F_1 in the sub-wff $(F_1 \& L)$ to *FALSE*, irrespective of L the whole sub-wff reduces to *FALSE*.

If $U' \cup \{U_1''\}$ is E-unsatisfiable, then it is a dual case of the above.

If $U' \cup \{U_1'\}$ gives a derivation of $\sim L$, then resolve it with the sub-wff $(F_1 \& L)$ when it is derived in R with the literal L added back, to give an NCRUE-NCNRF refutation of U.

Hence, we have got an NCRUE-NCNRF refutation of U, which has $n + 1$ excess literals.

Case(iii): (a) $F_1 \dashrightarrow L$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_1 \dashrightarrow L\} \cup \{\sim F_1\}$ and U_1'' as $U_1 - \{F_1\}$.

The proof is same as that for Case(i).

(b) $L \dashrightarrow F_1$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{L\}$ and U_1'' as $U_1 - \{L \dashrightarrow F_1\} \cup \{\sim L\}$.

Now the proof goes along the same lines of Case(i).

In both (iii) (a) and (b) the proof simply involves

rewriting $F_1 \dashrightarrow L$ and $L \dashrightarrow F_1$ as $\sim F_1 \vee L$ and $\sim L \vee F_1$ respectively.

Case(iv): $F_1 \longleftrightarrow L$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_1 \longleftrightarrow L\} \cup \{F_1 \longrightarrow L\}$ and U_1'' as $U_1 - \{F_1 \longleftrightarrow L\} \cup \{L \longrightarrow F_1\}$.

Since U is E-unsatisfiable, either of $U' \cup \{U_1'\}$ or $U' \cup \{U_1''\}$ or both are E-unsatisfiable.

Let us assume that $U' \cup \{U_1'\}$ is E-unsatisfiable. But, it has $n + 1$ excess literals. So, by the Case(iii)(a), there is an NCRUE-NCNRF refutation. This will give us an NCRUE-NCNRF refutation of U as well.

If $U' \cup \{U_1''\}$ is E-unsatisfiable, then by Case(iii)(b), there is an NCRUE-NCNRF refutation. This will give then an NCRUE-NCNRF refutation of U.

Hence we have got an NCRUE-NCNRF refutation of U, which has $n + 1$ excess literals.

Thus, we have proved the existence of an NCRUE-NCNRF refutation of U, which has $n + 1$ excess literals in all possible cases.

This completes the induction and proves that if U is ground and E-unsatisfiable, it has an NCRUE-NCNRF refutation. □

A variation of this proof by inducing on the number of binary connectives in the set U is given in Appendix I.

Theorem 4.8 Herbrand's Theorem

A set $U \cup E_U$ is unsatisfiable iff there is a finite unsatisfiable set, call it $U' \cup E'_{U'}$ of ground instances of wffs in $U \cup E_U$.

Proof: Refer to any of [ChL 73], [Lov 78]. □

Theorem 4.3 Completeness of NCRUE Resolution (Open form, General case)

If U is an E-unsatisfiable set of wffs, there exists an NCRUE-NCNRF deduction of *FALSE* from U.

Proof:

By Theorem 4.8, if U is E-unsatisfiable, there exists a substitution θ that instantiates U to a ground set U', which is E-unsatisfiable. U' may have multiple and distinct instantiations of the same wff in U. We have proved in Theorem 4.7 that U' has an NCRUE-NCNRF refutation which now serves as a refutation of U based on the substitution θ. Our proof is in open form, since we have not shown how to form θ. □

4.4 NCRUE Resolution in Strong Form

In this section, we wish to select a disagreement set and substitution that preserves completeness. Most of the definitions and examples are reproduced from [DiH 86].

Consider the following E-unsatisfiable set of wffs:

U: 1. $P(f(x, b))$

2. $\sim P(f(a, c))$

3. $f(b, b) = f(a, c)$

Using the *mgpu* (most general partial unifier) and the *bottommost disagreement set*, NCRUE Resolution resolves

$$
\begin{array}{lll}
P(f(x, b)) & & \\
\;\;\;| & & \\
\;\;\;|\text{-----} \;\; \sim P(f(a, c)) & & \theta\text{: } a/x \\
\;\;\;| & & \\
\sim(b = c) & &
\end{array}
$$

We cannot erase $\sim(b = c)$, because we cannot derive $b = c$ from the set U, and hence we cannot refute U.

From the above example, it is clear that the constant use of mgpu and bottommost disagreement set makes the NCRUE rules of inference incomplete, just as in the case of RUE Resolution. To refute the above, we must substitute b/x and resolve to the topmost disagreement set. Now, we will define a substitution and a disagreement set that are compatible with completeness.

4.4.1 Selection of a Disagreement Set

With respect to proving equality from an input clause set S, we have the following lemma, which is also true for the case of proving equality from an input set of wffs U.

Lemma 4.1 (On Proving Equality)

Given nonidentical terms, t_1, t_2, a necessary condition that $t_1 = t_2$ can be proved from $U \cup E_U$ is:

(1) $t_1 = t_2$ has the form $f(a_1,...,a_k) = f(b_1,...,b_k)$, where $a_i = b_i$, $i=1$ to k, can be proved from $U \cup E_U$, or

(2) there exists a transitivity chain in positive equality literals appearing in U:

$$t_1 \simeq r_1\theta = r_1'\theta \simeq r_2\theta = r_2'\theta \simeq ... \simeq r_k\theta = r_k'\theta \simeq t_2, \ k\geq 1,$$

where $r_i = r_i'$ (or $r_i' = r_i$), $i=1,2,...,k$, are literals of U instantiated by θ, and $\simeq$ denotes identity ($\equiv$) or equality ($=$). When the link $r_i'\theta \simeq r_{i+1}\theta$ is equality but not identity, these two terms must start with the same function symbol whose corresponding arguments can be proved equal from $U \cup E_U$.

Proof: Refer to [DiH 86]. □

The lemma has a recursive character that states that $t_1 = t_2$ can be proved from $U \cup E_U$ only by means of a set of one or more chains of the form defined in (2).

Note that in condition (2) the transitivity chain is purely in literals of U, which are linked by identity ($\equiv$) or the equality substitution axiom for functions. We have excluded a link by equality in which the substitution axiom for functions is not applied.

This lemma is a necessary but not sufficient condition because the equality literals of U used in condition (2)

may not appear as unit wffs in U and must be so derived before we can prove $t_1 = t_2$.

4.4.1.1 Definition (Viable Disagreement Set)

Using the Lemma 4.1, we can define a disagreement set to be *viable*, that is, can possibly participate in a refutation, only if it satisfies the following conditions:

(1) For each $\sim(s_i = t_i)$ in the disagreement set, we have that for s_i there is a term a that is the argument of a positive equality literal in U, such that s_i either unifies with a or matches a on leading function symbol. In the latter case, $s_i{:}a$ must have a viable disagreement set below the origin disagreement. The same is also true for t_i and some other term b that is the argument of a positive equality literal in U.

(2) In case there are some $\sim(s_i = t_i)$ that do not satisfy the above, then there is a substitution unifying s_i with t_i for each of these negative equalities, thus converting their logical sum to *FALSE*.

Furthermore, the substitutions used to satisfy the above condition must be compatible so as to form a single composite substitution. The empty disagreement set is always considered viable.

We introduce the equalities, specified by a disagreement set, as conditions (effectively negative equalities) into the resolvent at every resolution step. It is obvious

that such inequalities need to be erased, in order to derive a *FALSE* from any resolvent, by resolving each with a positive equality, either present in the initial set of wffs or deduced traversing through a transitivity chain among positive equalities. The viability check on disagreement set essentially ensures beforehand that such a positive equality indeed can be deduced. Thus, we will never generate a resolvent with inequalities that can not be erased. In other words, we generate only those resolvents that lead to a refutation. Thus, the viability criterion acts as a guidance in reasoning about equality.

The definition of viability is recursive, and the recursive iteration is finite because each of the terms, s_i and t_i, has a finite nesting of function symbols (the recursion peels these off and thus terminates).

Viability is a necessary but not sufficient condition that we can prove $s_i = t_i$ from U and thus erase the corresponding negative equality in a resolvent. An NCRUE-NCNRF resolvent is viable if its disagreement set is viable.

Consider the example

U: 1. $f(x) = b$ 5. $P(f(a))$

2. $b = c$ 6. $\sim P(g(d))$

3. $c = d$ 7. $\sim P(g(a))$.

4. $d = g(b)$

If we resolve $P(f(a))$ with $\sim P(g(d))$, we obtain $\sim(f(a) = g(d))$, which is viable since $f(a)$ unifies with $f(x)$ in 1 and $g(d)$ matches on function symbol with $g(b)$ in 4 with $\sim(d = b)$ being viable. We can in fact refute U by resolving $P(f(a))$ with $\sim P(g(d))$. On the other hand, if we resolve $P(f(a))$ with $\sim P(g(a))$, we obtain $\sim(f(a) = g(a))$, which is not viable. We cannot refute U by resolving 5 and 7.

Now note that in

U: 1. $P(f(a))$

2. $\sim P(f(b))$

3. $a = b$

we are saying that in resolving 1 and 2, the negative equality $\sim(f(a) = f(b))$ is not viable. Obviously, $\sim(f(a) = f(b))$ can participate in a refutation if we apply NCNRF to reduce it to $\sim(a = b)$, but by stating that it is not viable, we are forcing the theorem prover to go immediately to $\sim(a = b)$ and to suppress the appearance of $\sim(f(a) = f(b))$. When $\sim(s_i = t_i)$ is not viable, this means either that it cannot participate in a refutation because $s_i = t_i$ fails to satisfy the lemma or that the lemma requires in condition 1 that we proceed to lower level disagreements.

4.4.1.2 Rule for Selecting the Disagreement Set in NCRUE and NCNRF

In resolving $P^{+}(s_1,...,s_n)$ and $P^{-}(t_1,...,t_n)$ by NCRUE or in reducing a negative equality $t_1 =^{-} t_2$ by NCNRF, choose as D the topmost viable disagreement set.

We choose the topmost viable disagreement set as D because by using the NCNRF rule we can derive from D any lower level viable disagreement if it is needed in a refutation. When none of the disagreement sets in NCRUE and NCNRF are viable, then it is useless to form the resolvent since it cannot appear in a refutation. Our rule of selection will yield either one or no resolvent when we apply NCRUE and NCNRF. We will prove, in section 4.5, that this rule preserves completeness.

The weakness of the viability test lies in the fact that when U contains the literal $x = y$ or the literals $x = t$ and $y = t$, then all negative equalities become viable and the filtering effect is lost. In this case, we always resolve to the topmost disagreement set.

4.4.2 Selection of a Substitution

Since the use of the mgpu at each step of NCRUE-NCNRF can actually prevent a refutation, we now qualify the application of the mgpu. We state the rules of unification for NCRUE and NCNRF, respectively.

4.4.2.1 The NCRUE Unification Rule

In forming the unifier of complementary literals P^+, P^- in NCRUE, we unify on a variable x except at an occurrence where x is the argument of a function, say $f[x]$, such that there is a viable disagreement set of P^+, P^-, having a negative equality in the form $f[x] =^- f[t]$.

N.B.: $f[x]$ denotes one occurrence of x anywhere in the argument structure of f, and similarly for t in $f[t]$; t and x are at corresponding argument positions so that it is possible to unify x with t in P^+, P^-.

We may apply this rule to a specific occurrence of x without regard to any side effect on other occurrences of x in its clause. Consider the example given in [DiH 86],

U: 1. $P(f(x)) \vee Q(x) \vee R(g(x))$	4. $\sim R(g(c))$
2. $\sim P(f(a))$	5. $b = c$
3. $\sim Q(b)$	6. $f(a) = f(b)$

In resolving P and $\sim P$ we do not unify on x because it is the argument of a function and the functional disagreement $f(x){:}f(a)$ is viable without the unification. Here we resolve to

$$f(x) = f(a) \dashrightarrow [\, Q(x) \vee R(g(x))].$$

We are actually not foreclosing the unification a/x, which

can now be applied with NCNRF to $f(x) =^{-} f(a)$, if necessary. We are simply making it possible to explore the erasure of $f(x) =^{-} f(a)$ without this unification.

In the above set, our rule of unification permits us to unify on x in resolving Q with $\sim Q$ and R with $\sim R$, because in Q, x is the argument of a predicate, and in R, x does not participate in a viable functional disagreement.

A careful examination shows that the NCRUE unification rule produces only resolvents leading to a refutation in this set by permitting the substitutions b/x or c/x, but not a/x.

4.4.2.2 The NCNRF Unification Rule

When NCNRF is applied to $x =^{-} t$ we unify with t/x to erase this negative equality. In applying NCNRF to $f(a_1,\ldots,a_k) =^{-} f(b_1,\ldots,b_k)$ and forming the unifier of these opposed terms, we unify on a variable x except at an occurrence where x is the argument of an inner function (say, $g[x]$, which appears within the argument list of f), such that there is a viable disagreement set of $f(a_1,\ldots,a_k)$, $f(b_1,\ldots,b_k)$ having a negative equality of the form $g[x] =^{-} g[t]$.

N.B.: $g[x]$ denotes one occurrence of x anywhere in the argument list of g, and similarly for t in $g[t]$; x and t are at corresponding argument positions so that it is possible to unify x with t.

This is quite similar to the NCRUE unification rule

except that we state that x is the argument of an inner function and not of the outermost function f.

Consider again an example given in [DiH 86],

U: 1. $[\ f(g(x)) = f(g(c))\ \&\ f(x) = f(b)\ ] \dashrightarrow P(x)$

2. $g(a) = g(c)$

3. $a = b$

4. $\sim P(a)$.

In applying NCNRF to $f(g(x)) =^{-} f(g(c))$, we do not unify $g(x)$ and $g(c)$ because x is the argument of an inner function and the disagreement $g(x):g(c)$ is viable without unification. However, we do unify in applying NCNRF to $f(x) =^{-} f(b)$.

4.4.2.3 Rule for Selecting the Substitution in NCRUE-NCNRF

In resolving $P^{+}(s_1,...,s_n)$ with $P^{-}(t_1,...,t_n)$ by NCRUE or in reducing $t_1 =^{-} t_2$ by NCNRF, choose as θ the left-to-right mgpu-qualified by the NCRUE-NCNRF unification rules. We call this substitution *NCRUE Unifier of* P^{+}, P^{-} or t_1, t_2.

4.4.2.4 NCRUE Rule of Inference (Strong Form)

Let a wff U_i contain atoms $L_{i1},...,L_{in}$, such that θ_i is an NCRUE unifier of $\{L_{i1},...,L_{in}\}$ and D_i is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{L_{i1},...,L_{in}\}\theta_i = \{L'_{i1},...,L'_{ik}\}$; $1 \le k \le n$. Under the condition that D_i is

true, $\{L'_{i1},\ldots,L'_{ik}\} = \{L_i\}$.

Let another wff U_j contain atoms $L_{j1},\ldots,L_{jm}$, such that θ_j is an NCRUE unifier of $\{L_{j1},\ldots,L_{jm}\}$ and D_j is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{L_{j1},\ldots,L_{jm}\}\theta_j = \{L'_{j1},\ldots,L'_{jl}\}$; $1 \leq l \leq m$. If D_j is true, then $\{L'_{j1},\ldots,L'_{jl}\} = \{L_j\}$.

Let θ be an NCRUE unifier of $\{L_i, L_j\}$ and D be a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{L_i, L_j\}\theta = \{L'_i, L'_j\}$. If D is true, then $\{L'_i, L'_j\} = \{L\}$.

If L occurs positively in $U_i(\theta_i \bullet \theta) = U'_i$ and negatively in $U_j(\theta_j \bullet \theta) = U'_j$, then the *NCRUE resolvent* of U_i and U_j is

$$(D_i \ \& \ D_j \ \& \ D) \dashrightarrow (U'_i\{FALSE/L\} \vee U'_j\{TRUE/L\}).$$

If L is negative in U'_i and positive in U'_j, then we have the following dual NCRUE resolvent:

$$(D_i \ \& \ D_j \ \& \ D) \dashrightarrow (U'_i\{TRUE/L\} \vee U'_j\{FALSE/L\}).$$

We may use the rule for symmetry, defined in section 4.2.6, with NCRUE applied to equality literals.

4.4.2.5 NCNRF Rule of Inference (Strong Form)

Let a wff U_i contain equality atoms $=_1,...,=_n$, such that θ_i is an NCRUE unifier of $\{=_1,...,=_n\}$ and D_i is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{=_1,...,=_n\}\theta_i = \{='_1,...,='_k\}$; $1 \leq k \leq n$. If D_i is true, then $\{='_1,...,='_k\} = \{=\}$.

If $=$ has the form $t_1 = t_2$ and occurs negatively in $U_i\theta_i$, and θ is an NCRUE unifier of $\{t_1, t_2\}$ and D is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $t_1\theta$, $t_2\theta$, then the *Non-Clausal Negative Reflective Function (NCNRF) resolvent* of U_i is

$$(D_i \ \& \ D) \dashrightarrow U_i(\theta_i \bullet \theta)\{TRUE/t_1\theta = t_2\theta\}.$$

We shall reconsider the example given in section 4.2.8 to illustrate the rules of inference in strong form. The wffs are

U_1: $f(x, f(y, z)) = f(f(x, y), z)$

U_2: $f(a, f(b, f(c, d))) \neq f(f(f(a, b), c), d)$

The first resolution as given in section 4.2.8 conforms to the NCRUE unification rules. Thus, the resolvent R_1 is

$U_1 + U_2 \longrightarrow R_1$:

$\sim[f(b, f(c, d)) = f(y, d)\ \&\ f(f(a, b), c) = f(a, y)]$.

But, the symmetry variant of first resolution as given in section 4.2.8 is not viable though it conforms to the NCRUE unification rule and hence it will not take place. The disagreement set $\{f(y, f(b, f(c, d))):d, f(f(f(a, b), c), y):a\}$ is not viable because there is no positive equality in the set, such that a unifies with one of the arguments of that equality, and similarly for d also there is no positive equality, such that one of its arguments unifies with d.

In resolving R_1, U_1 using NCRUE resolution in open form, we can use the substitution $\{b/x_1, c/y_1, d/z_1, f(b, c)/y\}$. But, it does not conform to the NCRUE unification rule, as c is an argument of inner function in $f(b, f(c, d))$. Hence we will use $\{b/x_1, d/z_1\}$ and the disagreement set $\{f(c, d):f(y_1, d)\}$ to generate the resolvent

$U_1 + R_1 \longrightarrow R_2$:

$f(c, d) = f(y_1, d) \longrightarrow \sim(f(f(a, b), c) = f(a, f(b, c)))$

Now NCNRF rule can be applied to the sub-wff, the antecedent of the implication in R_2. With the substitution $\{c/y_1\}$ we get

$R_2 \longrightarrow_{NCNRF} R_3$: $\sim(f(f(a, b), c) = f(a, f(b, c)))$

Again we are not permitted to unify $f(b, c)$, $f(y, z)$ and $f(a, b)$, $f(x, y)$ in order to resolve U_1, R_3 under NCRUE unification. Hence, we use $\{a/x, c/z\}$ and disagreement set $\{f(a, b){:}f(a, y), f(b, c){:}f(y, c)\}$ to generate

$$U_1 + R_3 \dashrightarrow R_4: \sim[f(a, b) = f(a, y) \;\&\; f(b, c) = f(y, c)]$$

Applying NCNRF to R_4 with the substitution $\{b/y\}$ on equality $f(a, b) = f(a, y)$ we get

$$R_4 \dashrightarrow_{NCNRF} R_5: \sim(f(b, c) = f(b, c))$$

which can be reduced to *FALSE* by one more NCNRF application.

$$R_5 \dashrightarrow_{NCNRF} R_6: FALSE.$$

For the *Group Theory* problem illustrated in section 4.2.8, the first three resolutions conform to the strong form of NCRUE resolution.

In the fourth resolution we unified $g(g(a))$, $g(x)$ with $\{g(a)/x\}$ where these two are arguments to a function in respective wffs. Since this substitution is not allowed under NCRUE unification rules, we get a different resolvent.

$R_3 + U_3 \dashrightarrow R_4$: $g(g(a)) = g(y) \rightarrow \sim(f(y, a) = e)$;{a/x}

$R_4 \dashrightarrow_{NCNRF} R_5$: $\sim(f(g(a), a) = e)$;{g(a)/y}

$R_5 + U_3 \dashrightarrow R_6$: FALSE ;{}

4.4.3 The Equality Restriction

For further efficiency we extend the equality restriction, imposed on the RUE resolution of complementary equality literals [DiH 86], to NCRUE.

The NCRUE resolution of $U_i(s_1 =^+ s_2)$ and $U_j(t_1 =^- t_2)$ is permitted only if at least one pair in the set of pairs $\{s_1{:}t_1, s_1{:}t_2, s_2{:}t_1, s_2{:}t_2\}$ unifies or matches on leading-function-symbol. In the latter case, the pair must have a viable disagreement set below the origin disagreement.

Hence, to resolve equality literals we must satisfy both the above equality restrictions and be able to select a topmost viable disagreement set as previously defined. These two conditions appear similar but they are not the same. In finding a topmost viable disagreement of the complementary literals $s_1 = s_2$, $t_1 =^- t_2$, the matching by unification or matching on leading-function-symbol occurs with other literals of U.

Consider the E-unsatisfiable set:

U: 1. $\sim(a = b)$ 4. $e = c$

2. $c = d$ 5. $b = f$

3. $a = e$ 6. $f = d$.

$\sim(a = b)$ should not be resolved with $c = d$ because the equality restriction is not satisfied even though the resolvent $\sim(a = c \ \& \ b = d)$ is viable. Out of the five candidates that resolve with $\sim(a = b)$, only two satisfy the equality restriction, namely, $a = e$ and $b = f$. The NCRUE refutation of U can be achieved within the constraints of the equality restriction.

The completeness of the equality restriction is proved in section 4.5.

4.5 Completeness Results for NCRUE Resolution in Strong Form

In this section we prove the completeness of the topmost viable disagreement set, the completeness of the equality restriction, and based on these completeness results state a conjecture for the completeness of NCRUE resolution in strong form.

Lemma 4.2 Completeness of the Topmost Viable Disagreement Set

If U is E-unsatisfiable, there exists an NCRUE refutation of U that uses the topmost viable disagreement set at each deduction step.

Proof:

Since U is E-unsatisfiable, by completeness there

exists an NCRUE refutation of U, call it R.

The viability criterion is a direct application of Lemma 4.1, which states a necessary condition for proving equality in U. If a disagreement set is not viable, it contains a negative equality, $a =^{-} b$, whose complement $a = b$ does not satisfy condition (2) of Lemma 4.1 and, hence, this equality cannot be derived from U which means that $a =^{-} b$ cannot be erased. The only way of erasing $a =^{-} b$ in a refutation is by traversing transitivity chains that derive $a = b$ from U, and these do not exist when $a =^{-} b$ is not viable.

Hence, a disagreement set appearing in a deduction step of R must be viable, that is, it can only contain viable negative equalities.

If D, a disagreement set in a deduction step of R, is not the topmost viable disagreement set, then we can modify this deduction step to use the topmost viable disagreement set by introducing NCNRF steps to descend to D, that is, we replace

$$U_i\langle L^{+}(s_1,\dots,s_n)\rangle$$

$$|\text{-----}\ U_j\langle L^{-}(t_1,\dots,t_n)\rangle$$

$$D \dashrightarrow [U_i\theta\{FALSE/L(s_1,\dots,s_n)\theta\} \vee U_j\theta\{TRUE/L(t_1,\dots,t_n)\theta\}]$$

by

$$
\begin{array}{l}
\qquad\qquad\qquad U_i\langle L^+(s_1,\ldots,s_n)\rangle \\
\qquad\qquad\qquad\qquad\qquad |\text{-----}\ \ U_j\langle L^-(t_1,\ldots,t_n)\rangle \\
D' \longrightarrow [U_i\theta\{FALSE/L(s_1,\ldots,s_n)\theta\} \vee U_j\theta\{TRUE/L(t_1,\ldots,t_n)\theta\}] \\
\quad \textit{topmost viable disagreement set} \\
\qquad\qquad\qquad\qquad \vdots \qquad \textit{NCNRFs using the topmost viable disagreement set} \\
D \longrightarrow [U_i\theta\{FALSE/L(s_1,\ldots,s_n)\theta\} \vee U_j\theta\{TRUE/L(t_1,\ldots,t_n)\theta\}]
\end{array}
$$

The same is done with any NCNRF step in R that does not use the topmost viable disagreement set.

This modification procedure can be applied to any NCRUE refutation of U so that every refutation has a canonical form using the topmost viable disagreement set.

In respect to the equality restriction, derived from Lemma 4.1, we have the following Lemma 4.3 which leads to its completeness, stated by Lemmas 4.4 & 4.5 for the ground and general cases respectively.

Lemma 4.3 (On the Equality Restriction)

If U_E is a set of positive unit equalities such that $U_E \cup \{\sim(t_1 = t_2)\}$ is E-unsatisfiable, then there exists a linear-input refutation of this set that satisfies the equality restriction and has the form

$\sim(t_1 = t_2)$

(NCNRF)

$\vert$ - - - - - L_1

R_1

$\vert$ - - - - - L_2

R_2

.

.

.

R_{k-1}

$\vert$ - - - - - L_k

FALSE

where $L_i \in U_E$, $i=1,2,...,k$, and θ is being applied as the refutation substitution. *(NCNRF)* denotes a possibly required NCNRF step applied to $\sim(t_1 = t_2)$.

Proof: Almost similar to its clausal counterpart, given in [DiH 86]. □

Lemma 4.4 Completeness of the Equality Restriction (Ground Case)

If U is an E-unsatisfiable ground set, there exists an NCRUE refutation of U that satisfies the equality restriction.

Proof:

Let $k(U)$ be defined to be the total number of appearances of literals in U minus the number of wffs in U. Our proof is by induction on $k(U)$, the *excess literal parameter*.

(1) Suppose $k(U) = \emptyset$, U is a set of unit wffs. Choose any negative wff U_i in U that we assume is minimally E-inconsistent.

(a) If U_i is a negative equality $\sim(t_1 = t_2)$, since U is minimally E-inconsistent it must have the form $U = \{\sim(t_1 = t_2)\} \cup U_E$, where U_E is a set of positive unit equalities. Hence, by Lemma 4.3, there is an NCRUE refutation of U that satisfies the equality restriction.

(b) If U_i is a nonequality literal, $\sim P(t_1,\ldots,t_n)$, then U must contain its complement $P(s_1,\ldots,s_n)$ and we can resolve

$$P(s_1,\ldots,s_n) \;\&\; \sim P(t_1,\ldots,t_n) \;\text{---}\!> D,$$

the topmost viable disagreement set.

Since U is a minimally E-inconsistent unit set with $k(U) = \emptyset$, if D is not empty, it follows that $U - \{P(s_1,\ldots,s_n), \sim P(t_1,\ldots,t_n)\}$ is a set of unit positive equalities U_E and that $U_E \cup \{D\}$ is E-unsatisfiable. Let $D = \{\sim(t_i = t_i'), i=1,2,\ldots,k\}$. We can apply Lemma 4.3 to $\{\sim(t_1 = t_1')\} \cup U_E$ to obtain R_1, which satisfies the equality restriction. The concatenation of the R_i, $i=1,2,\ldots,k$, then forms a refutation R of $U_E \cup \{D\}$. Hence, if we begin with $P(s_1,\ldots,s_n) \;\&\; \sim P(t_1,\ldots,t_n) \;\text{---}\!> D$ and apply R to D, we have a refutation that refutes U and satisfies the equality restriction.

We have shown that when $k(U) = \emptyset$, there is a refutation of U that satisfies the equality restriction.

(2) Now suppose our Lemma 4.4 holds for U ground and $k(U) \leq m$. We show that it holds for $k(U) = m + 1$.

Suppose ground U has $m + 1$ excess literals. Consider a wff U_1 of U having an excess literal L. Depending on how L is effectively connected to other sub-wff, say F_1, in U_1, we will have 4 different cases.

Case(i): $F_1 \vee L$

Define U' as $U - \{U_1\}$ and U_1' as $U_1 - \{L\}$, and U_1'' as $U_1 - \{F_1\}$. Since U is E-unsatisfiable, so are $U' \cup \{U_1'\}$ and $U' \cup \{U_1''\}$. Since both of these have m or fewer excess literals, they both have NCRUE refutations that satisfy the equality restriction.

Now consider the refutation R_1 of $U' \cup \{U_1'\}$. Modify R_1 by adding back the literal L to the wff U_1'. This may give an NCRUE refutation of U in which case we derive a *FALSE* without the participation of L. Otherwise, this will give an NCRUE derivation of L from U. By appending to this derivation the portion of the refutation of $U' \cup \{U_1''\}$ where L is refuted, we get an NCRUE refutation, that satisfies the equality restriction, of ground U, which has $m + 1$ excess literals.

Case(ii): F_1 & L

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{L\}$ and U_1'' as

$U_1 - \{F_1\}$. Since U is E-unsatisfiable, either $U' \cup \{U_1'\}$ or $U' \cup \{U_1''\}$ is E-unsatisfiable, or both, or at least $U' \cup \{U_1'\}$ will give a derivation of $\sim L$.

Let us assume that $U' \cup \{U_1'\}$ is E-unsatisfiable. Since it has m or fewer excess literals, there is an NCRUE refutation that satisfies the equality restriction. This will give us an NCRUE refutation, satisfying the equality restriction, of U as well, because when we reduce F_1 in the sub-wff $(F_1 \& L)$ to $FALSE$, irrespective of L the whole sub-wff reduces to $FALSE$.

If $U' \cup \{U_1''\}$ is E-unsatisfiable, then it is a dual case of the above.

If $U' \cup \{U_1'\}$ gives a derivation of $\sim L$, then resolve it with the sub-wff $(F_1 \& L)$ when it is derived in R with the literal L added back, to give an NCRUE refutation of U that satisfies the equality restriction.

Hence, we have got an NCRUE refutation of U, which has $m + 1$ excess literals, satisfying the equality restriction.

Case(iii): $(a)\ F_1 \dashrightarrow L$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_1 \dashrightarrow L\} \cup \{\sim F_1\}$ and U_1'' as $U_1 - \{F_1\}$.

The proof is the same as that for Case(i).

$(b)\ L \dashrightarrow F_1$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{L\}$ and U_1'' as

$U_1 - \{L \rightarrow F_1\} \cup \{\sim L\}$.

Now the proof goes along the same lines of Case(i).

Case(iv): $F_1 \leftrightarrow L$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_1 \leftrightarrow L\} \cup \{F_1 \rightarrow L\}$ and U_1'' as $U_1 - \{F_1 \leftrightarrow L\} \cup \{L \rightarrow F_1\}$.

Since U is E-unsatisfiable, either $U' \cup \{U_1'\}$ and $U' \cup \{U_1''\}$ is E-unsatisfiable or both.

Let us assume that $U' \cup \{U_1'\}$ is E-unsatisfiable. But, it has $m + 1$ excess literals. So, by the case(iii)(a), there is an NCRUE refutation that satisfies the equality restriction. This will give us an NCRUE refutation of U as well that satisfies the equality restriction.

If $U' \cup \{U_1''\}$ is E-unsatisfiable, then by Case(iii)(b), there is an NCRUE refutation. This will give then an NCRUE refutation of U satisfying the equality restriction.

Hence we have got an NCRUE refutation that satisfies the equality restriction of U, which has $m + 1$ excess literals.

Thus, we have proved the existence of an NCRUE refutation that satisfies the equality restriction of U, which has $m + 1$ excess literals in all possible cases.

This completes the induction to prove that if U is ground and E-unsatisfiable

satisfies the equality restriction. □

Lemma 4.5 Completeness of the Equality Restriction (General Case)

> If U is an E-unsatisfiable set of wffs, there exists an NCRUE refutation of U that satisfies the equality restriction.

Proof:

Since U is E-unsatisfiable, $U \cup E_U$ is unsatisfiable and, by Herbrand's Theorem, there is a finite unsatisfiable set $U' \cup E_U'$ of ground instances of wffs in $U \cup E_U$. Hence, the ground set U' is E-unsatisfiable. By Lemma 4.4, there exists a ground NCRUE refutation of U' that satisfies the equality restriction and serves to refute U.

Completeness of the equality restriction has been proven in open form since we have not specified the substitution θ which instantiates U to U' and is the basis for a refutation of U satisfying the equality restriction. □

To illustrate the role of the equality restriction consider the E-unsatisfiable set

U: 1. $\sim(a = b)$	4. $e = c$
2. $c = d$	5. $b = f$
3. $a = e$	6. $f = d$.

chain

$$a = e \approx e = c \approx c = d \approx d = f \approx f = b.$$

The equality restriction requires that ~(a = b) traverse this chain from endpoint to endpoint (see the figure below). In the second refutation, we are traversing the same chain of equalities, but we enter the chain at an interior point and work outward to each endpoint.

```
~(a = b)        as opposed to,      ~(a = b)
   |            for example              | (does not
   |                                     |  satisfy
   |                                     |  equality
   |                                     |  restriction)
   |----- a = e                          |----- c = d
   |                                     |
~(b = e)                        ~(a = c & b = d)
   |                                     |
   |----- c = e                          |----- e = c
   |                                     |
~(b = c)                        (a = e) --> ~(b = d)
   |                                     |
   |----- d = c                          |----- a = e
   |                                     |
~(b = d)                             ~(b = d)
   |                                     |
   |----- f = d                          |----- f = d
   |                                     |
~(b = f)                             ~(b = f)
   |                                     |
   |----- b = f                          |----- b = f
   |                                     |
 FALSE                                 FALSE
```

Conjecture 4.1 Completeness of NCRUE Resolution in Strong Form

The completeness of the NCRUE-NCNRF rules of inference is preserved if at each application of these inference rules we

(1) choose as θ, the NCRUE unifier,

(2) choose as D, the topmost viable disagreement set,

(3) satisfy the equality restriction when resolving complementary equality literals.

Support for Conjecture:

We have proved in Lemmas 4.2 and 4.5 that if U is E-unsatisfiable there exists a refutation that uses the topmost viable disagreement set and applies the equality restriction. Hence, we have proved completeness in respect to parts (2) and (3). It yet remains to prove that a refutation exists that satisfies the NCRUE-NCNRF unification rules.

In [Dig 83] Digricoli has defined a transformation procedure that, given a refutation of S, an E-unsatisfiable set of clauses, that satisfies properties (2) and (3) and is based on a composite substitution θ, transforms θ so that it conforms to the RUE-NRF unification rules. Though there is no guarantee that this procedure terminates, it yields on termination a refutation that satisfies all three properties stated for RUE resolution. We feel that this transformation procedure should easily be extendible to the non-clausal case, thus providing more justification for our conjecture. □

4.6 NCRUE Resolution on sub-wffs

We will first define the notion of a disagreement set with respect to two complementary sub-wffs, and then give the rules of inference.

4.6.1 A Disagreement Set of Complementary Sub-wffs

A disagreement set of complementary sub-wffs, F_1, F_2 where F_1 contains the literals $P_1^+(s_{11},...,s_{1n})$ $,...,P_k^+(s_{k1},...,s_{kn})$ and F_2 contains the complementary literals $P_1^-(t_{11},...,t_{1n}),...,$ $P_k^-(t_{k1},...,t_{kn})$, is defined as the union

$$D = \bigcup_{i=1}^{k} D_i,$$

where D_i is a disagreement set of the corresponding complementary literals $P_i^+(s_{i1},...,s_{in})$, $P_i^-(t_{i1},...,t_{in})$.

The topmost disagreement set of F_1, F_2 is the union of topmost disagreement sets of corresponding complementary literals.

Using the substitution axiom for every predicate symbol, we can state that

$$F_1 \;\&\; F_2 \longrightarrow D$$

where D now represents the negation of a conjunction of the equalities to be satisfied as specified by any disagreement set of F_1, F_2. In NCRUE we can resolve F_1 and F_2 immediately to D.

For example, by NCRUE we may resolve

```
P(f(a, h(b, g(c)))) op Q(i(j(a), k(b)))
                     |
                     |
                     |
                     |-----  ~[P(f(b, h(c, g(d))))
                     |        op Q(i(j(d), k(e)))]
                     |
                     |
                     D
```

in 4 * 3 distinct ways depending on our choice of D. We may resolve to $\sim(f(a, h(b, g(c))) = f(b, h(c, g(d))) \;\&\; i(j(a), k(b)) = i(j(d), k(e)))$, to $\sim(a = b \;\&\; b = c \;\&\; c = d \;\&\; a = d \;\&\; b = e)$, or to an intermediate level D. So, if, in a sub-wff, there are k distinct literals having maximum depths of $n_1, \ldots, n_k$, then there are $n_1 * n_2 * \ldots * n_k$ distinct ways of choosing D.

Similarly with respect to NCNRF, the sub-wff containing only negative equalities

$$f(a, h(b, g(c))) =^- f(b, h(c, g(d))),$$
$$i(j(a), k(b)) =^- i(j(d), k(e)) \longrightarrow D$$

reduces in 3 * 2 different ways depending on our choice of D.

4.6.2 NCRUE Rule of Inference (Strong Form) to include resolution on sub-wffs

Let a wff U_i contain sub-wffs $F_{i1}, \ldots, F_{in}$, such that θ_i is an NCRUE unifier of $\{F_{i1}, \ldots, F_{in}\}$ and D_i is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_{i1}, \ldots, F_{in}\}\theta_i = \{F'_{i1}, \ldots, F'_{ik}\}$; $1 \le k \le n$. Under the condition that D_i is

true, $\{F'_{i1},\ldots,F'_{ik}\} = \{F_i\}$.

Let another wff U_j contain sub-wffs $F_{j1},\ldots,F_{jm}$, such that θ_j is an NCRUE unifier of $\{F_{j1},\ldots,F_{jm}\}$ and D_j is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_{j1},\ldots,F_{jm}\}\theta_j = \{F'_{j1},\ldots,F'_{jl}\}$; $1 \leq l \leq m$. If D_j is true, then $\{F'_{j1},\ldots,F'_{jl}\} = \{F_j\}$.

Let θ be an NCRUE unifier of $\{F'_i, F_j\}$ and D be a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_i, F_j\}\theta = \{F'_i, F'_j\}$. If D is true, then $\{F'_i, F'_j\} = \{F\}$.

If F occurs positively in $U_i(\theta_i \circ \theta) = U'_i$ and negatively in $U_j(\theta_j \circ \theta) = U'_j$, then the *NCRUE resolvent* of U_i and U_j is

$$(D_i \ \& \ D_j \ \& \ D) \dashrightarrow (U'_i\{FALSE/F\} \vee U'_j\{TRUE/F\}).$$

If F is negative in U'_i and positive in U'_j, then we have the following dual NCRUE resolvent:

$$(D_i \ \& \ D_j \ \& \ D) \dashrightarrow (U'_i\{TRUE/F\} \vee U'_j\{FALSE/F\}).$$

We may use the rule for symmetry, defined in section 4.2.6, with NCRUE applied to equality literals.

4.6.3 NCNRF Rule of Inference (Strong Form) to include resolution on sub-wffs

Let a wff U_i contain sub-wffs $F_1,\ldots,F_n$, such that each F_j contains only positive equality literals and θ_i is an NCRUE unifier of $\{F_1,\ldots,F_n\}$, and D_i is a conjunction of equalities to be satisfied as specified by the top-most viable disagreement set of $\{F_1,\ldots,F_n\}\theta_i = \{F'_1,\ldots,F'_k\}$; $1 \leq k \leq n$. If D_i is true, then $\{F'_1,\ldots,F'_k\} = \{F\}$.

If F contains equalities of the form $t_{j1} = t_{j2}$ and occurs negatively in $U_i\theta_i$, and θ_j is an NCRUE unifier of $\{t_{j1}, t_{j2}\}$ and all θ_j's are compatible to form a composite substitution θ, and D is a conjunction of equalities to be satisfied as specified by the union of the top-most viable disagreement sets of $t_{j1}\theta$, $t_{j2}\theta$, $j=1, \mid F \mid$; then the *Non-Clausal Negative Reflective Function (NCNRF) resolvent of* U_i is

$$(D_i \ \& \ D) \dashrightarrow U_i(\theta_i \circ \theta)\{TRUE/F\theta\}.$$

For instance, reconsider the example presented in section 4.2.8 and its refutation in strong form. In the resolvent

$U_1 + R_3 \dashrightarrow R_4$:

$\sim[f(a, b) = f(a, y) \ \& \ f(b, c) = f(y, c)]$

we can apply NCNRF rule for wffs, as the wff contains only negative equalities. With the substitution $\{b/y\}$ we get

$$R_4 \dashrightarrow_{NCNRF} R_5: FALSE.$$

Now, we shall see that this generalization of NCRUE resolution preserves completeness.

4.6.4 Completeness of NCRUE resolution on sub-wffs

The alternate proof for the completeness of NCRUE resolution, given in Appendix I, also serves as a proof for the completeness of NCRUE resolution on sub-wffs.

The following Lemma 4.6 together with the completeness of NCRUE resolution, also proves that the NCRUE resolution on sub-wffs is complete.

Lemma 4.6

> In a refutation of an E-unsatisfiable set U of wffs, if two wffs U_i and U_j containing respectively the complementary sub-wffs F_i and F_j participate in successive resolutions, each time resolving upon a literal in these sub-wffs, to eliminate F_i, F_j, then this sequence of resolutions can be *subsumed* into a single resolution on sub-wffs F_i and F_j.

Proof:

Let us assume that the sub-wffs F_i and F_j respectively contain atoms $L_{i1},...,L_{in}$, and $L_{j1},...,L_{jn}$, and

generating the resolvent

$$R: U_i\theta\{FALSE/F_i\theta\} \vee U_j\theta\{TRUE/F_j\theta\}$$

when F_i is positive in U_i and F_j is negative in U_j.

In the former resolution, the final resolvent R_n contains many repetitions like $U_{i1} \vee \ldots \vee U_{i1}$, essentially so many times as the wff U_i has participated in resolution, where $U_{i1} = U_i\theta - F_i\theta$. Similarly, it contains repetitions of sub-wffs like $U_{j1} \vee \ldots \vee U_{j1}$ so many times as the wff U_j has participated in resolution, where $U_{j1} = U_j\theta - F_j\theta$.

All such repeated instances of sub-wffs would essentially be identical and hence can be merged to make R_n equivalent to R. Thus, a resolution sequence for eliminating literals in two complementary sub-wffs, one literal at a time, can be subsumed into a single resolution on the sub-wffs. Hence, the lemma. □

Thus, NCRUE resolution on sub-wffs is also a sound and complete proof procedure.

Chapter 5

MANY-SORTED AND EQUALITY-BASED NON-CLAUSAL RESOLUTION

We have seen the advantages of Many-sorted NC-resolution in Chapter 3, and those of Equality-Based NC-resolution in Chapter 4. By extending NCRUE resolution to include sorting, we can take advantage of MNC-resolution as well as NCRUE resolution.

In this chapter, we describe how sort information can be strengthened by exploiting *unit equalities* present in the set and how Many-sorted NCRUE resolution (MNCRUE resolution) is performed on MS sentences. We also show that the soundness and completeness properties of MNC-resolution and NCRUE resolution are preserved in MNCRUE resolution.

5.1 Introduction

Schmidt [Sch 85] has given a rule based mechanism for converting an unsorted set of clauses into its many-sorted version. Jaya Raghavendra [Jay 87] has given a set of just 3 rules by which an unsorted set of wffs can be converted into its many-sorted version. We have added an additional rule (Chapter 3) to it without affecting the completeness. But

this set of rules does not take advantage of the unit equalities present in the set of wffs.

We make use of the *Rules* EQ_i of [Sch 85], appropriately modified to conform to the sort theory of chapter 3, to strengthen the sortedness in view of extending many-sorted theory to NCRUE resolution; and at the same time, preserving the motivation behind Jaya Raghavendra's work to have a few simple rules by which we can get most of the sort information.

5.2 Sorting Rules for Equality

After we get the many-sorted version of a set of wffs U by the algorithm as described in Chapter 3, we repeatedly apply the following rules until there are no changes in the sort information, in two successive applications.

Rule EQ_1:

If there is a positive unit equality, $c = t$, where c is a constant or a function symbol of sort S_c and t is a term of sort S_t, then the sort of c is changed into $S_c \cap S_t$.

Rule EQ_2:

If there is a positive unit equality, $x = t$, where x is a variable of sort S_x and t is a term of sort S_t, then a sort axiom $S_x \subseteq S_t$ is added.

We use the symmetry property of equality, if necessary, while applying these rules.

5.3 Many-sorted NCRUE resolution

The *many-sorted non-clausal resolution by unification and equality (MNCRUE resolution)* is the NCRUE resolution (Chapter 4) with *MS unification*. The substitutions made in NCRUE resolution should be sort compatible as defined in Chapter 3.

The notion of disagreement sets is the same as in the case of NCRUE resolution. All the concepts of the strong form of NCRUE resolution, such as topmost viable disagreement set, NCRUE unification, and equality restriction, extend to MNCRUE resolution as well, with the substitution or unification being an *MS substitution* or *MS unification*. We state here the rule for selecting the substitution in MNCRUE-MNCNRF deduction and the rules of inference directly in strong form for resolution on sub-wffs.

5.3.1 Rule for selecting the substitution in MNCRUE-MNCNRF deduction

In resolving $P^{+}(s_1,\ldots,s_n)$ with $P^{-}(t_1,\ldots,t_n)$ by MNCRUE or in reducing $t_1 =^{-} t_2$ by MNCNRF, choose as θ the left-to-right MS mgpu-qualified by the MNCRUE-MNCNRF unification rules. We call this substitution *MNCRUE Unifier* of P^{+}, P^{-} or t_1, t_2.

5.3.2 MNCRUE Rule of Inference (Strong Form) to include resolution on sub-wffs

Let a wff U_i contain sub-wffs $F_{i1},\ldots,F_{in}$, such that θ_i is an MNCRUE unifier of $\{F_{i1},\ldots,F_{in}\}$ and D_i

is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_{i1},\ldots,F_{in}\}\theta_i = \{F'_{i1},\ldots,F'_{ik}\}$; $1 \leq k \leq n$. Under the condition that D_i is true, $\{F'_{i1},\ldots,F'_{ik}\} = \{F_i\}$.

Let another wff U_j contain sub-wffs $F_{j1},\ldots,F_{jm}$, such that θ_j is an MNCRUE unifier of $\{F_{j1},\ldots,F_{jm}\}$ and D_j is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_{j1},\ldots,F_{jm}\}\theta_j = \{F'_{j1},\ldots,F'_{jl}\}$; $1 \leq l \leq m$. If D_j is true, then $\{F'_{j1},\ldots,F'_{jl}\} = \{F_j\}$.

Let θ be an MNCRUE unifier of $\{F_i, F_j\}$ and D be a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_i, F_j\}\theta = \{F'_i, F'_j\}$. If D is true, then $\{F'_i, F'_j\} = \{F\}$.

If F occurs positively in $U_i(\theta_i \circ \theta) = U'_i$ and negatively in $U_j(\theta_j \circ \theta) = U'_j$, then the *MNCRUE resolvent* of U_i and U_j is

$$(D_i \,\&\, D_j \,\&\, D) \dashrightarrow (U'_i\{FALSE/F\} \vee U'_j\{TRUE/F\}).$$

If F is negative in U'_i and positive in U'_j, then we have the following dual MNCRUE resolvent:

$$(D_i \,\&\, D_j \,\&\, D) \dashrightarrow (U'_i\{TRUE/F\} \vee U'_j\{FALSE/F\}).$$

We may use the rule for symmetry, defined in section

4.2.6, with MNCRUE applied to equality literals.

5.3.3 MNCNRF Rule of Inference (Strong Form) to include resolution on sub-wffs

Let a wff U_i contain sub-wffs $F_1, \ldots, F_n$, such that each F_j contains only positive equality literals and θ_i is an MNCRUE unifier of $\{F_1, \ldots, F_n\}$, and D_i is a conjunction of equalities to be satisfied as specified by the topmost viable disagreement set of $\{F_1, \ldots, F_n\}\theta_i = \{F_1', \ldots, F_k'\}$; $1 \leq k \leq n$. If D_i is true, then $\{F_1', \ldots, F_k'\} = \{F\}$.

If F contains equalities of the form $t_{j1} = t_{j2}$ and occurs negatively in $U_i\theta_i$, and θ_j is an MNCRUE unifier of $\{t_{j1}, t_{j2}\}$ and all θ_j's are compatible to form a composite substitution θ, and D is a conjunction of equalities to be satisfied as specified by the union of the topmost viable disagreement sets of $t_{j1}\theta$, $t_{j2}\theta$, $j=1, |F|$; then the *Many-sorted Non-Clausal Negative Reflective Function (MNCNRF) resolvent of* U_i is

$$(D_i \;\&\; D) \longrightarrow U_i(\theta_i \circ \theta)\{TRUE/F\theta\}.$$

Since MNC-resolution is not complete without the *Weakening rule*, as we have seen in chapter 3, the MNCRUE resolution will also not be complete without it. The definition of Weakening rule in respect to MNCRUE resolution is the same as that for MNC-resolution.

Let us consider a small example to understand how the sorting rules for equality affect the sort information and how the many-sorted NCRUE rules of inference work.

The following set of wffs is E-unsatisfiable.

U_1: $R(b)$

U_2: $D(x) \dashrightarrow L(a, x)$

U_3: $R(y) \dashrightarrow (Q(z) \dashrightarrow \sim L(y, z))$

U_4: $Q(a) \dashrightarrow D(a)$

U_5: $Q(a)$

U_6: $a = b$

If we do not use the sorting rules for equality in converting the above set into a many-sorted set, then the MS version is

Sort axioms: $Q \subseteq D$

Sorts of terms: $x{:}D$, $y{:}R$, $z{:}Q$,

$a{:}Q$, $b{:}R$.

MS sentences:

M_1: $L(a, x)$

M_2: $\sim L(y, z)$

M_3: $a = b$.

In order to resolve the wffs M_1, M_2, we have to unify the sequences of terms (a, x), (y, z) under sorting. The substitution $\{z/x\}$ is sort compatible where as $\{x/z\}$ is

not. The substitution $\{a/y\}$ is not sort compatible. But, the disagreement set $\{a:y\}$ is a viable disagreement set under sorting, i.e. there is a positive equality in the set such that a MS unifies with one of the arguments and similarly for y also. Hence, we can apply MNCRUE resolution to M_1, M_2 to generate the MNCRUE resolvent

$$M_1 + M_2 \dashrightarrow R_1 : \sim(a = y) \quad ;\{z/x\}$$

Now, we can resolve R_1 with M_3 to generate *FALSE* thus refuting the given set.

$$M_3 + R_1 \dashrightarrow R_2 : FALSE \quad ;\{b/y\}$$

If we make use of the sorting rules for equality in converting the unsorted set into a many-sorted set, then the previously obtained MS set will get modified to the following.

Sort axioms: $Q \subseteq D$, $\quad Q \cap R \subseteq Q$,

$Q \cap R \subseteq R$, $\quad Q \cap R \subseteq D$.

Sorts of terms: $x:D$, $\quad y:R$, $\quad z:Q$,

$a:Q \cap R$, $\quad b:Q \cap R$.

MS sentences:

M_1: $L(a, x)$

M_2: $\sim L(y, z)$

M_3: $a = b$.

Now in a single resolution between M_1 and M_2, we can refute the set because the substitution *{z/x, a/y}* is sort compatible.

$$M_1 + M_2 \dashrightarrow R_1: FALSE.$$

Agatha's problem, originally given by Schubert and taken from [Pel 86], is solved in Chapter 10 using MNCRUE resolution.

5.4 Soundness and Completeness of MNCRUE resolution

Just as the soundness and completeness proofs of MNC-resolution [Jay 87] make use of the properties of NC-resolution [Mur 82], we also show that MNCRUE resolution is sound and complete, based on the properties of NCRUE resolution (Chapter 4). The MNCRUE resolution is assumed to be in the form, either *open* or *strong,* corresponding to that of NCRUE resolution, in the following proofs.

If we don't make use of the sorting rules for equality, as given in section 5.2, while converting an unsorted set of wffs into its many-sorted version, then the proofs would be just the same as those for MNC-resolution, with NC-resolution replaced by NCRUE resolution and MNC-resolution by MNCRUE resolution. That is so because the equality predicate is a *two-place predicate* (i.e., just like any other predicate) which does not change the sort information in any manner.

If we wish to make use of the sorting rules for equality, which will strengthen the sortedness, then the proofs would differ a little from those for MNC-resolution. We will first develop the background for proofs as in [Jay 87] and, then give soundness and completeness proofs in that order.

5.4.1 Overview

Let U be an unsorted set of wffs, and M its many-sorted version. U can be considered to be the union of two sets of wffs U_a and U_b where

$U_a = \{ U_{ai} \mid U_{ai}$ is a wff in U, with at least one atom whose predicate is not a sort in M $\}$

$U_b = \{ U_{bi} \mid U_{bi}$ is a wff in U, with atoms all of whose predicates are sorts in M $\}$

Clearly, $M = \{M_{a1}, M_{a2}, \ldots\}$ where M_{ai} is the MS version of U_{ai}. The translation of U_b in MS goes as sort information and does not appear in the sorted formulae. In the following proofs, let MR denote a many-sorted inference step, either an application of an MNCRUE rule of inference or MNCNRF rule of inference, or Weakening rule; and UR denote an inference step in US, either an application of NCRUE rule of inference or NCNRF rule of inference.

Let us assume that $FALSE$ can be derived from M, by a linear sequence of MR, $MR_1, \ldots, MR_k$, where MR_k gives the resolvent $FALSE$. There is a sequence of one or more

UR corresponding to every MR. We say that a linear sequence of UR corresponds to an MNCRUE resolution, if every atom that participates in it, participates in a UR of the sequence.

A single MNCRUE resolution may correspond to more than one UR, when the resolution is on sub-wffs. Similarly, a weakening rule application may also correspond to a sequence of one or more UR.

Let $UR_1,\ldots,UR_n$, $n \geq k$, be the sequence of UR, corresponding to $MR_1,\ldots,MR_k$. UR_n gives the resolvent R_n corresponding to $FALSE$ in MR. So UR_n should be a wff with only predicates which are sorts in MS. The soundness theorem proves that $FALSE$ can be deduced from such a R_n. Thus it is proved that, all sets of wffs from which $FALSE$ is deduced through MR, are indeed unsatisfiable (or, $FALSE$ can be deduced from them in US using UR). This proves the soundness of our method.

Now consider the case where $FALSE$ was deduced from U, through a linear sequence of UR. This sequence of UR can be arranged as $UR_1,\ldots,UR_n,\ldots,UR_m$, where all UR_i, $i \leq n$, involve wffs from U_a, and all UR_i, $i > n$, involve wffs from U_b. Lemma 5.6 proves that $FALSE$ cannot be deduced from the set of wffs U_b alone. Hence, at least one $U_{ai} \in U_a$ should be involved in deriving $FALSE$ from U. So there will be at least one MNCRUE resolution in the corresponding sequence of MR. Every UR_i, $i \leq n$, has a corresponding linear sequence of MR. For such a sequence

of MR to go through in MS, all unifications in $UR_1, \ldots, UR_n$ should be possible in MS. In MS, a substitution may be prevented because of sort incompatibility. The completeness theorem proves that, if a sort incompatibility were to occur for even one such substitution in MS, then it wouldn't have been possible to derive *FALSE* from R_n. Since *FALSE* was derived from R_n, we conclude that no sort incompatibility arises. In other words, *FALSE* can also be deduced in MS. Thus the system is complete.

5.4.2 Soundness

We prove a few results before taking up the problem of soundness.

Lemma 5.1

For any two sorts P and Q, if $P \subseteq Q$ then one of the following is true.

(i) the wff $P(x) \dashrightarrow Q(x)$ for some variable x, was either present in US or could have been deduced.

(ii) the positive unit equality, $x = t$, for some variable x and some term t of sorts P and Q respectively, was either present or could have been deduced.

Proof:

Case (i): Refer to the proof of Lemma 1 in [Jay 87].

Case (ii): If $P \subseteq Q$ is a sort axiom, then $x = t$ for some variable x and some term t was present in US; else $P \subseteq Q$ was inferred from the sort axioms using the rules of

section 3.2. The first rule is equivalent to the addition of reflexive axiom $x = x$, and the second rule is an application of the transitivity axiom for equality. These two are the only rules applicable to the deduction of an equality from a given set of wffs. Hence if $P \subseteq Q$ was inferred from those rules, then $x = t$ can be deduced in US. □

Lemma 5.2

If P is a sort in MS, then the wff $P(c)$ where c is a constant or a function is either present or can be deduced.

Proof:

A unary predicate can become a sort in MS, iff there is at least one object of that sort, or its subsorts. In the former case, $P(c)$ for some constant or function c, representing that object will be explicitly present in US. Otherwise, $Q(c)$ exists where $Q \subseteq P$. Then from Lemma 5.1 Case (i), $Q(x) \;\text{-->}\; P(x)$ for some variable x either exists or can be deduced, and a resolution gives $P(c)$. From Case (ii), an equality $x = t$ for some variable x of sort Q and some term t of sort P either exists or can be deduced. But, from the sorting rules for equality, it is ascertained that both P and Q are already valid sorts in MS, while introducing the sort axiom $Q \subseteq P$. Hence, the deduction of such a wff $P(c)$ must have been only in the way as Case (i). □

Lemma 5.3

For a sort P and a constant or a function c, $[c] \subseteq P$ iff one of the following is true.

(i) the wff $P(c)$ can be deduced in US.

(ii) the wff $P(c)$ or $P(d)$ where d is another constant or function, such that $c = d$ is a positive equality in US, can be deduced in US.

Proof:

Case (i): Refer to the proof of Lemma 3 in [Jay 87].

Case (ii): If P were a sort in MS added by the rules based on equality, then these rules ensure that a positive unit equality, $c = d$, is either present or could have been deduced through a transitivity chain among positive equalities, and also ensure that P is already a sort of c or d. Hence, a $P(c)$ or $P(d)$ should be in US, or have been deduced, which can be explained by Lemma 5.2. □

Theorem 5.1 (Soundness Theorem)

If M is the many-sorted translation of the set of wffs U, then

$$M \overset{MR^*}{\vdash} FALSE \longrightarrow U \overset{UR^*}{\vdash} FALSE$$

Proof:

Let us arrange the sequence of resolutions in MS and US as explained in section 5.4.1. Now it remains to be proved that $FALSE$ can be derived from R_n, the resolvent

corresponding to the resolvent *FALSE* in MS. R_n will be a wff with only s.forms from U_a that have participated in the earlier NCRUE resolutions. In MS, these s.forms would have been removed by the use of general rules 1, 2 and 4 only, and the sorting rules for equality do not remove any s.forms. R_n can't be having the s.forms that get removed by rule 3. This is because in a rule 3 application, the sub-wff F_1 & F_2 should occur positively in the wff, and a resolution on F_2 reduces this to *FALSE*.

So R_n is of the form

$$R_n: \sim S_1(t_1) \vee \ldots \vee \sim S_k(t_k)$$

where S_i is an s.form, and t_i is a term such that $[t_i] \subseteq S_i$.

A point to note is that for some i, $i = 1,k$, $[t_i]$ may not be S_i itself. This is because of earlier unifications where t_i was substituted in place of some variable x_i, as $[t_i] \subseteq [x_i]$. In any case, $[t_i] \subseteq S_i$.

Now based on the terms t_i, we divide R_n into sub-wffs of the following three kinds as in the soundness proof given in [Jay 87], and prove that each of them can be eliminated from R_n, without bothering about the rest of R_n.

(a) Sub-wffs of the form $\sim S_i(t_i)$ where t_i is a variable such that for no j, $j = 1,k$, $(j \neq i)$ $t_i = t_j$.

(b) Sub-wffs of the form $\sim S_i(c_i)$ where c_i is a constant

or a function.

(c) Sub-wffs of the form $\sim S_{i1}(t_i) \vee \ldots \vee \sim S_{i1}(t_i)$, where t_i is a variable.

All s.forms represent sorts and we ensure that there is at least one element of every sort. From Lemma 5.2, $S_i(c_i)$ where c_i is a constant or a function is present in US or can be deduced. So sub-wffs of the kind (a) can be resolved with such $S_i(c_i)$.

The sub-wffs of the other two kinds should have come into R_n due to earlier unifying substitutions. Let us now consider sub-wffs of the kind (b). If $[c_i]$ is S, then $S(c_i)$ should be present in US. For some c_i, $[c_i]$ can be S_i itself, because some sub-wffs of the kind (b) might have been removed in MS due to the application of rule 4. In any case, $[c_i] \subseteq S_i$. So, from Lemma 5.3, $S_i(c_i)$, or $S_i(d_i)$ such that $c_i = d_i$ is an equality in US, can be deduced. In either case, $\sim S_i(c_i)$ can be eliminated from R_n.

In sub-wffs of the kind (c), the sort of the variable t_i should be one of the S_{ij}'s. Let $[t_i] = S_{im}$. From Lemma 5.2, $S_{im}(c)$, where c is a constant or a function, is derivable. $\sim S_{im}(t_i)$ can be resolved with $S_{im}(c)$ using the substitution $\{c/t_i\}$. Now for all j, $[c] \subseteq S_{ij}$. So every $\sim S_{ij}(c)$, $j \neq m$, is a sub-wff of the kind (b) which can be eliminated.

So *FALSE* can be deduced from R_n, or in other words, if we are able to derive *FALSE* in MS using *MR*, then we

can be sure that *FALSE* can be derived in US using *UR*. Hence, *MR* is a sound proof procedure. □

5.4.3 Completeness

We first establish some lemmas needed for the completeness proof.

Lemma 5.4

If P is a sort in MS, then the wff $P(x)$ where x is a variable, can neither exist nor be deduced in US.

Proof: Refer to the proof of Lemma 4 in [Jay 87]. □

Lemma 5.5

If P is a sort in MS, then the wff $\sim P(x)$ where x is a variable, can neither exist nor be deduced in US.

Proof: Refer to the proof of Lemma 5 in [Jay 87]. □

Lemma 5.6

The set of wffs U_b is E-satisfiable.

Proof:

Since there are no equalities present in the set U_b, the proof for the satisfiability of U_b in Lemma 6 of [Jay 87] would serve as a proof for its E-satisfiability as well. □

Theorem 5.2 (Completeness Theorem)

If M is the many-sorted translation of the set of wffs U, then

$$U \overset{UR^*}{\longmapsto} FALSE \dashrightarrow M \overset{MR^*}{\longmapsto} FALSE$$

Proof:

The set of wffs U_b has been proved to be E-satisfiable (Lemma 5.6). So at least one UR involving a U_{ai} is needed to deduce $FALSE$ from U. Arrange this sequence of UR as $UR_1,\ldots,UR_n,\ldots,UR_m$ where UR_i, $i \leq n$, involve wffs from U_a, and UR_i, $i > n$, involve wffs from U_b. Every UR_i, $i \leq n$, has corresponding to it, a MNCRUE resolution, or a weakening rule application and a MNCRUE resolution, such that the resolved-upon sub-wff is the same as that in UR_i.

Let $MR_1,\ldots,MR_k$, $k \geq n$, be the corresponding sequence of MR, where MR_k gives the resolvent $FALSE$. Now we prove that all unifications necessary for this sequence of MR can be done.

Let $\{t_1/t_2\}$ be a unifying substitution in a UR_i, $i \leq n$, which is prevented in MS due to sort incompatibility. Then $[t_1] \nsubseteq [t_2]$. If there is a term t, such that t unifies with t_1 as well as t_2, then we apply the weakening rule using the substitution $\{t/t_1\}$, and then perform a MNCRUE resolution using $\{t/t_2\}$. In case no such term t exists and $\{t_1/t_2\}$ has been used in UR_i, $i \leq n$, then that UR_i cannot be in the sequence of UR that derives $FALSE$ from U. This is because if such a UR_1 were to be in the sequence of UR, then UR_n would result in the

resolvent R_n having a sub-wff $\sim S_i(t_i)$ where $[t_i] \nsubseteq S_i$. Such a sub-wff cannot be resolved-upon as it has been proved in Lemma 5.3 that for any sort S_i, $S_i(c_i)$ or $S_i(d_i)$ such that $c_i = d_i$ is a positive equality in US, can be derived in US, iff $[c_i] \subseteq S_i$. Since *FALSE* was derived from R_n, no sort incompatibility can occur in MS for all substitutions in $UR_1,\ldots,UR_n$. Hence, *FALSE* can be deduced from M in MS. Thus MR is complete. □

Thus the MNCRUE resolution is sound and complete.

RESOLUTION FOR WELL-FORMED FORMULAE

The NC-resolution proposed by Murray [Mur 82] requires the wffs to be quantifier free. Although this addresses the disadvantages of Robinson's clausal resolution [Rob 65] which we have discussed in Chapter 2, both of these resolution principles still have the following inherent disadvantages:

(1) The intuition behind selecting appropriate quantifiers in expressing the problem is lost in the conversion.

(2) The sentence becomes too complex when quantifiers are removed, in particular those with in the scope of (nested) equivalences.

The second disadvantage may well be illustrated by the following example.

Let a wff be

$$U_1: [(\forall x)\ (\exists y)\ A(x, y) \longleftrightarrow (\forall z)\ (\exists u)\ B(z, u)] \\ \longleftrightarrow [(\exists v)\ (\forall w)\ C(v, w) \longleftrightarrow (\forall s)\ (\exists t)\ D(s, t)]$$

When converted into a *standard normal form* as a step

towards making it quantifier free, this generates, out of the only 8 variables present, 8 universal and 16 existential variables adding to 24. In addition to this, it will then have the following grisly form.

$$(\forall x)\ (\exists y_1)\ (\exists z_1)\ (\forall u)\ (\exists x_1)\ (\forall y)\ (\forall z)\ (\exists u_1)$$
$$(\forall v)\ (\exists w_1)\ (\forall s)\ (\exists t_1)\ (\exists v_1)\ (\forall w)\ (\exists s_1)\ (\forall t)$$
$$(\exists y_2)\ (\exists z_2)\ (\exists x_2)\ (\exists u_2)\ (\exists w_2)\ (\exists t_2)\ (\exists v_2)\ (\exists s_2)$$
$$[(\{\ (A(x,\ y_1)\ \longrightarrow B(z_1,\ u))\ \&\ (B(z,\ u_1)\ \longrightarrow A(x_1,\ y))\ \}$$
$$\longrightarrow \{\ (C(v,\ w_1)\ \longrightarrow D(s,\ t_1))\ \&\ (D(s_1,\ t)\ \longrightarrow C(v_1,\ w))\ \})$$
$$\&\ (\{\ (C(v_2,\ w)\ \longrightarrow D(s_2,\ t))\ \&\ (D(s,\ t_2)\ \longrightarrow C(v,\ w_2))\ \}$$
$$\longrightarrow \{\ (A(x_2,\ y)\ \longrightarrow B(z,\ u_2))\ \&\ (B(z_2,\ u)\ \longrightarrow A(x,\ y_2))\ \})]$$

In regard to these problems, Manna & Waldinger [MaW 82] have dealt with a non-clausal deductive system applied to sentences that may have *some* of their quantifiers intact. In an attempt to overcome the above disadvantages and to preserve the syntactic form of a wff, such that the proofs become more readable and human oriented, we have developed a non-clausal deductive system for sentences containing *all* the quantifiers intact. That is to say precisely that we have developed a deduction system for wffs in FOL.

6.1 Introduction

The reader should refer to [Sko 20] or [Sko 28] for a full treatment of what are called *Skolem functions,* and to [Qui 61] for *Quantification theory.* In our methodology we dispense with Skolem functions which otherwise are

substituted for existential variables. Instead, we talk about dependencies of an existential variable, which will be defined shortly.

We will first define the notion of a term in our system which emphasizes upon the fact that a variable can be *existential* as well.

6.1.1 Definition (Terms)

Terms are defined recursively as follows:

(i) A constant is a term.

(ii) A universal variable is a term.

(iii) An existential variable is a term whenever its dependencies (see section 6.1.3 for definition) are terms.

(iv) If f is an n-place function symbol, and $t_1,...,t_n$ are terms, then $f(t_1,...,t_n)$ is a term.

(v) All terms are generated by applying the above rules.

We are taking into account *constants* and *functions* since we may have explicit occurrences of constants and functions that are different from Skolem constants and Skolem functions.

The notions, such as an atom, a literal and a wff are the same as given in Chapter 2. The definition of polarity (Chapter 2) for a wff also extends to the definition of polarity for quantifiers, and can be stated as below.

6.1.2 Definition (Polarity of a Quantifier)

Let Q be a quantifier, S_1 and S_2 be wffs.

(1) If Q is positive (negative) in S_1, then it is negative (positive) in $\sim S_1$ and $(S_1 \dashrightarrow S_2)$,

(2) If Q is positive (negative) in S_1, then it is positive (negative) in $(S_1 \& S_2)$, $(S_1 \vee S_2)$ and $(S_2 \dashrightarrow S_1)$,

(3) If Q occurs in S_1, then it is both positive and negative in $(S_1 \leftarrow\!\!\dashrightarrow S_2)$.

In making a wff quantifier free, a quantifier Q, present in the wff, passes through a set of negations depending on which it will remain as Q or its *complement* in the prefix of Skolem standard form of the wff. If it passes through an even number of negations, then it remains as Q and otherwise it transforms into its complement. The definition of polarity essentially captures the number of negations within whose scope a quantifier occurs. Hence, by taking the quantification of a term in a wff depending on the polarity of the quantifier, we get the same quantification as it would be in the Skolem standard form of the wff. Since resolution procedure does not change the polarity of any constituent of a wff in forming the resolvent, the quantification of a term, determined in the beginning by the polarity of the quantifier, does not change during resolution.

The other notions of Chapter 2, such as Structural

equivalence and complementarity also carry over to our system.

6.1.3 Definition (Dependent Existential Variable)

An existential variable in a wff is a *dependent existential variable,* if at least one universal quantifier precedes it. The *dependency information (or dependencies)* of such a dependent variable, analogous to the arguments of Skolem functions, is a list of all universally quantified variables preceding it.

For example, in the wff $(\forall x)\ [P(x) \dashrightarrow (\exists y)\ Q(x,\ y)]$, y is a dependent existential variable and its dependency information is (x).

6.1.4 Definition (Independent Existential Variable)

An existential variable in a wff is an *independent existential variable,* if it's not dependent.

For example, in the wff $(\exists x)\ (\forall y)\ (P(x)\ \&\ Q(x,\ y))$, x is an independent existential variable.

We shall introduce the notion of a dual quantifier which is different from the already existing notion of quantification [Qui 61].

6.1.5 Definition (Dual Quantifier)

A quantifier Q that is both positive and negative in a wff is a *dual quantifier.* In other words, a quantifier Q that occurs within the scope of an equivalence is a *dual quantifier.*

For example, in $[(Qx)\ P(x) \longleftrightarrow R]$ where Q is a quantifier, Q is dual. It can act as both universal and existential depending on the polarity chosen. The variable x is said to be *dually quantified.* When Q is positive, it is Q, and when it is negative, it is Q^c, the complement of Q.

This chapter describes an algorithm for obtaining dependency information of an existential variable in a wff without paraphrasing the wff in any manner and with quantifiers in place. This also introduces the concept of unification in the context of quantified variables, which we call *Q-unification,* and describes how to perform WFF-resolution. The algorithm together with the WFF-resolution is proved to be sound and complete by showing that it is equivalent to quantifier free NC-resolution.

6.2 Obtaining Dependencies

We shall now describe the *algorithm* to obtain dependency information (dependencies).

Intuitively, to obtain the dependencies of an existential instance of a variable in a wff, our algorithm scans the wff from left-to-right and prepares a list of universally quantified variables occurring in it. For any existential variable, the dependencies are the list of universal variables so far obtained.

We assume that any variable in the given set of wffs is uniquely named, and that each wff is properly parenthesized.

The processing is recursive in nature; if the connective is binary, then each sub-wff is processed separately and the dependencies are modified accordingly; if the connective is unary (i.e. ~), then the only sub-wff is processed by interchanging the notions of universal and existential quantifications.

The algorithm in LISP notation is given below and a trace of the functioning of the algorithm for two wffs, one with equivalences and the other without, is also given for illustrative purposes.

```
(setq algofns '(dependencies get-deps rem-props print-deps is-atf
               is-qtfr is-uqtfr opr modify-deps
               dlistu dliste toggle merge set-union first second))

(defun dependencies(f flag1 flag2)
           ;;; function implementing the Algorithm to obtain dependencies
           ;;; for a wff 'f'.
           ;;; all the variables in f are assumed to be properly renamed
           ;;; and f is properly parenthesized.
           ;;; flag1, if TRUE then f is positive else negative in the
           ;;; wff in which it occurs.
           ;;; flag2, if TRUE then f is within the scope of equivalence
           ;;; else not, in the wff in which it occurs.

     (cond[(is-atf f) '(() ())]
           ;;; if f is atomic formula without quantifiers, then there
           ;;; are no universal or existential variables in f. return
           ;;; a list of two null lists, since the list of universal
           ;;; and existential variables is null

          [(and (is-qtfr (car f)) flag2)
           ;;; if there is a dual quantifier in the prefix of f, then
           ;;; do the following.

              (let ((var (cadar f))
                    (dlists (dependencies (cdr f) flag1 flag2)))
           ;;; obtain the dlists for (cdr f), that is f with the
           ;;; first quantifer removed.

                   (modify-deps var (dliste dlists))
           ;;; modify the dependencies of all existential variables in
           ;;; (cdr f) by including var.
           ;;; include the var in dlists as described below.

                   (cond[(is-uqtfr (car f) t)
           ;;; if the var is universally quantified as it appears, then

                         (list (cons var (dlistu dlists)) (dliste dlists))]
           ;;; add this var to the universal variable list of dlists.

                        [t (list (dlistu dlists) (cons var (dliste dlists)))]
           ;;; else, add this var to the existential variable list of dlists.
                   ))]

          [(is-qtfr (car f))
           ;;; if there is a quantifier in the prefix of f, then
           ;;; do the following.

              (let ((var (cadar f))
                    (dlists (dependencies (cdr f) flag1 flag2)))
           ;;; obtain the dlists for f with the first quantifier removed.
           ;;; now, based on the type of quantifier, add this var to dlists
           ;;; of (cdr f) and/or modify dependencies of (cdr f).
```

```
        (cond[(is-uqtfr (car f) flag1)
;;; if the quantifier is universal taking polarity into account,
;;; then we need to modify the dependencies of (cdr f).

                (modify-deps var (dliste dlists))
                (list (cons var (dlistu dlists)) (dliste dlists))
;;; add this var to the universal variable list of dlists.
              ]

              [t (list (dlistu dlists) (cons var (dliste dlists)))
              ]
;;; else, we just add it to the existential variable list of dlists
        ))]

[(eq '~ (opr f))
;;; if the main connective of f is negation, then the notions
;;; of universal and existential variables interchange in f.

    (dependencies (cadr f) (toggle flag1) flag2)]
;;; obtain the dependencies of f with ~ removed, changing the
;;; notions of quantification.

[(or (eq 'V (opr f)) (eq '& (opr f)))
;;; if the main connective of f is either disjunction or
;;; conjunction, then

    (let ((dl1 (dependencies (first f) flag1 flag2))
          (dl2 (dependencies (second f) flag1 flag2)))
;;; obtain the dependencies of each sub-wff in f separately.

        (let ((ulist (dlistu dl1)) (elist (dliste dl2)))
             (modify-deps ulist elist)
;;; modify the dependencies of all existential variables in
;;; second sub-wff of f to include the universal variables of
;;; first sub-wff of f.

             (merge dl1 dl2)
;;; combine the dlists of sub-wffs of f into one and return it.
        ))]

[(eq '-> (opr f))
;;; if the main connective is implication, then

    (let ((dl1 (dependencies (first f) (toggle flag1) flag2))
;;; obtain the dependencies of first sub-wff of f changing
;;; the notions of quantification in it, because the first sub-wff
;;; is negative in f.

          (dl2 (dependencies (second f) flag1 flag2)))
;;; obtain the dependencies of second sub-wff.

        (let ((ulist (dlistu dl1)) (elist (dliste dl2)))
             (modify-deps ulist elist)
;;; modify the dependencies of all existential variables in
;;; second sub-wff of f to include the universal variables of
```

```
                (merge dl1 dl2)
  ;;; combine dlists of the two sub-wffs into one and return it.
              ))]

   [(eq '<-> (opr f))
    ;;; if the main connective is equivalence, then

       (let ((dl1 (dependencies (first f) flag1 t))
             (dl2 (dependencies (second f) flag1 t)))
    ;;; obtain dependencies of both the sub-wffs.

             (modify-deps (dlistu dl1) (dlistu dl2))
    ;;; modify the dependencies of all universal variables in
    ;;; second sub-wff of f to include the universal variables of
    ;;; first sub-wff of f, because the sub-wffs are within the
    ;;; scope of equivalence.

             (modify-deps (dliste dl1) (dliste dl2))
    ;;; modify the dependencies of all existential variables in
    ;;; second sub-wff of f to include the existential variables of
    ;;; first sub-wff of f, because of the same reason.

             (let ((dl (merge dl1 dl2)))
    ;;; combine dlists of both the sub-wffs.

                    (let ((l (set-union (dlistu dl) (dliste dl))))
                         (list l l))
    ;;; since all the variables in f are dual, combine the lists of
    ;;; universal variables and existential variables, and make this
    ;;; both universal variable list and existential variable list
    ;;; of new dlist for f.
             ))]

    [(eq (length f) 1)
     ;;; if more than one pair of parentheses enclose f, remove the
     ;;; the extra pair of parentheses and obtain the dependencies.

        (dependencies (car f) flag1 flag2)]
    [t (print '(Error - Inappropriate input format)) (exit)]
))


(defun is-atf(f)
        ;;; checks whether f is a quantifier free atomic formula.

    (cond[(null f) t]
         [(atom f) t]
         [(is-qtfr (car f)) ()]
         [(eq (length f) 1) ()]
         [(null (opr f)) t]
         [t ()]))
```

```
(defun is-qtfr(q)
       (cond[(atom q) ()]
           [(eq (car q) 'A) t]
           [(eq (car q) 'E) t]
           [t ()]))


(defun is-uqtfr(q sign)
       (cond[sign (eq (car q) 'A)]
           [t (eq (car q) 'E)]))


(defun opr(f)
       (cond[(atom f) ()]
           [(and (eq (length f) 2) (eq (car f) '~)) (car f)]
           [(and (eq (length f) 3) (member (cadr f) '(V & -> <->))) (cadr f)]
           [t ()]))


(defun modify-deps(varl evarl)
       (cond[(null evarl) ()]
           [(null varl) ()]
           [t (let ((val (get (car evarl) 'deps)))
                  (putprop (car evarl) (cond[(atom varl) (cons varl val)]
                                             [t (append varl val)])
                                       'deps)
                  (modify-deps varl (cdr evarl)))]))


(defun dlistu(dl)
       (car dl))

(defun dliste(dl)
       (cadr dl))

(defun toggle(flag)
       (cond[flag ()]
           [t t]))

(defun merge(dl1 dl2)
       (list (append (dlistu dl1) (dlistu dl2))
             (append (dliste dl1) (dliste dl2))))

(defun set-union(l1 l2)
       (cond[(null l1) l2]
           [(null l2) l1]
           [(member (car l2) l1) (set-union l1 (cdr l2))]
           [t (set-union (append l1 (list (car l2))) (cdr l2))]))

(defun first(f)
       (car f))

(defun second(f)
       (caddr f))
```

```
5.dskin algo
[load algo]
(algo)
6.dskin inputs
[load inputs]
(inputs)
7.setq $Vars ()
nil
8.(trace dependencies)
(dependencies)
9.pp f1
(setq f1
      '((((A x) (((A y) P) <-> ((E z) Q))) <-> ((A u) R)) V (* ((A v) S))))
t
10.(get-deps f1)
1 <Enter> dependencies ((((& &) <-> (& R)) V (* (& S))) t nil)
|2 <Enter> dependencies ((((A x) (& <-> &)) <-> ((A u) R)) t nil)
| 3 <Enter> dependencies (((A x) ((& P) <-> (& Q))) t t)
| |4 <Enter> dependencies ((((& P) <-> (& Q))) t t)
| | 5 <Enter> dependencies ((((A y) P) <-> ((E z) Q)) t t)
| | |6 <Enter> dependencies (((A y) P) t t)
| | | 7 <Enter> dependencies ((P) t t)
| | | |8 <Enter> dependencies (P t t)
| | | |8 <EXIT>  dependencies  (nil nil)
| | | 7 <EXIT>  dependencies  (nil nil)
| | |6 <EXIT>  dependencies  ((y) nil)
| | |6 <Enter> dependencies (((E z) Q) t t)
| | | 7 <Enter> dependencies ((Q) t t)
| | | |8 <Enter> dependencies (Q t t)
| | | |8 <EXIT>  dependencies  (nil nil)
| | | 7 <EXIT>  dependencies  (nil nil)
| | |6 <EXIT>  dependencies  (nil (z))
| | 5 <EXIT>  dependencies  ((y z) (y z))
| |4 <EXIT>  dependencies  ((y z) (y z))
| 3 <EXIT>  dependencies  ((x y z) (y z))
| 3 <Enter> dependencies (((A u) R) t t)
| |4 <Enter> dependencies ((R) t t)
| | 5 <Enter> dependencies (R t t)
| | 5 <EXIT>  dependencies  (nil nil)
| |4 <EXIT>  dependencies  (nil nil)
| 3 <EXIT>  dependencies  ((u) nil)
|2 <EXIT>  dependencies  ((x y z u) (x y z u))
|2 <Enter> dependencies ((* ((A v) S)) t nil)
| 3 <Enter> dependencies (((A v) S) nil nil)
| |4 <Enter> dependencies ((S) nil nil)
| | 5 <Enter> dependencies (S nil nil)
| | 5 <EXIT>  dependencies  (nil nil)
| |4 <EXIT>  dependencies  (nil nil)
| 3 <EXIT>  dependencies  (nil (v))
|2 <EXIT>  dependencies  (nil (v))
1 <EXIT>  dependencies  ((x y z u) (x y z u v))
dependencies of x: nil
dependencies of y: (x)
dependencies of z: (x)
dependencies of u: (x y z)
```

```
11.pp f2
(setq f2
      '((A x) (((E y) ((S x y) & (M y))) -> ((E z) ((I z) & (EN x z))))))
t
12.(get-deps f2)
1 <Enter> dependencies (((A x) ((& &) -> (& &))) t nil)
|2 <Enter> dependencies ((((& &) -> (& &))) t nil)
| 3 <Enter> dependencies ((((E y) (& & &)) -> ((E z) (& & &))) t nil)
| |4 <Enter> dependencies (((E y) ((S x y) & (M y))) nil nil)
| | 5 <Enter> dependencies ((((S x y) & (M y))) nil nil)
| | |6 <Enter> dependencies (((S x y) & (M y)) nil nil)
| | | 7 <Enter> dependencies ((S x y) nil nil)
| | | 7 <EXIT>  dependencies  (nil nil)
| | | 7 <Enter> dependencies ((M y) nil nil)
| | | 7 <EXIT>  dependencies  (nil nil)
| | |6 <EXIT>  dependencies  (nil nil)
| | 5 <EXIT>  dependencies  (nil nil)
| |4 <EXIT>  dependencies  ((y) nil)
| |4 <Enter> dependencies (((E z) ((I z) & (EN x z))) t nil)
| | 5 <Enter> dependencies ((((I z) & (EN x z))) t nil)
| | |6 <Enter> dependencies (((I z) & (EN x z)) t nil)
| | | 7 <Enter> dependencies ((I z) t nil)
| | | 7 <EXIT>  dependencies  (nil nil)
| | | 7 <Enter> dependencies ((EN x z) t nil)
| | | 7 <EXIT>  dependencies  (nil nil)
| | |6 <EXIT>  dependencies  (nil nil)
| | 5 <EXIT>  dependencies  (nil nil)
| |4 <EXIT>  dependencies  (nil (z))
| 3 <EXIT>  dependencies  ((y) (z))
|2 <EXIT>  dependencies  ((y) (z))
1 <EXIT>  dependencies  ((x y) (z))
dependencies of z: (x y)
nil
13.exit
```

Given the wff

U_1: $(\forall x)$ $[(\exists y)$ $(S(x, y)$ & $M(y))$ --> $(\exists z)$ $(I(z)$ & $EN(x, z))]$

as input, our algorithm generates the following dependencies.

$dep(z)$: (x, y).

If we consider the same textual order of sub-wffs, then the Skolem standard form of U_1 is

$(\forall x)$ $(\forall y)$ $(\exists z)$ $[(S(x, y)$ & $M(y))$ --> $(I(z)$ & $EN(x, z))]$

and the dependencies are

$dep(z)$: (x, y)

which are same as those obtained by our algorithm.

Since there are no equivalences and quantifiers within the scope of them in the input wff, we obtained the same dependencies as in Skolem standard form. Let us consider the following wff to illustrate how they would be different when there are equivalences in the wff.

Given the wff

U_2: $[\{$ $(\forall x)$ $[(\forall y)$ A <--> $(\exists z)$ $B]$ $\}$ <--> $(\forall u)$ C $]$ V $\sim(\forall v)$ D

as input to our algorithm, it gives out the dependencies of existential variables as below.

$dep(x)$: ()

$dep(y)$: (x)

$dep(z)$: (x)

$dep(u)$: (x, y, z)

$dep(v)$: (x, y, z, u).

Some of the dependencies of an existential instance of a dually quantified variable are being missed out by this algorithm. For example, the Skolem standard form of the above wff is

$$(\forall x_1)\ (\exists y_1)\ (\exists z_1)\ (\forall y)\ (\forall z)\ (\forall u)$$
$$(\forall x)\ (\exists y_2)\ (\exists z_2)\ (\exists u_2)\ (\exists v)$$
$$[\{((P \longrightarrow Q)\ \&\ (Q \longrightarrow P)) \longrightarrow R\}$$
$$\&\ \{R \longrightarrow ((P \longrightarrow Q)\ \&\ (Q \longrightarrow P))\}]$$

in which the existential instance of the dual variable u is dependent on its universal instance. Not only this but also the dual variables y, z each has two different existential instances (y_1, y_2 and z_1, z_2) having different dependencies as given below.

$dep(x_1)$: ()

$dep(z_1)$: ( ) $dep(z_2)$: (y, z, u, x)

$dep(u_1)$: (y, z, x, u)

$dep(v)$: (y, z, x, u).

But our algorithm ignores such dependencies and it can be justified as below.

An observation on Murray's work [Mur 82] of NC-resolution reveals the fact that in a refutation, we use only one part of the two implications that an equivalence gives rise to. Henceforth the two implications which make up an equivalence will be called *implicants* for stylistic reasons. From this we can assert that in a refutation we use only one instance, either universal or existential, of a dual variable. This can be illustrated by the following example.

Let us consider two wffs

U_1: $(\forall x)\ A(x) \longleftrightarrow (\forall y)\ B(y)$

U_2: $(\forall z)\ (\ A(z) \vee C(z))$

In resolving U_1, U_2 on the atom A, we consider the polarity of A in U_1 to be negative, which otherwise is dual. Accordingly, we take the polarity of the quantifier. Then the resolvent of U_1, U_2 would be

$[TRUE \longleftrightarrow (\forall y)\ B(y)] \vee (\exists x)\ [\ FALSE \vee C(x)\ ]$;$\{(\exists x)/(\forall z)\}$

which simplifies to

$$(\forall y)\ B(y)\ V\ (\exists x)\ C(x).$$

When we do such a resolution, the universal instance of x will not appear in the resolvent simply because we are not generating the other implicant of the equivalence. So it is not necessary to consider those dependencies of an existential variable which get generated from the implicant of the equivalence that does not participate in resolution.

We need not consider such dependencies, because & is commutative and we can assume that the implicant which participates in resolution always precedes the other implicant. That is the precise reason why the existential variable x, in the above example, is not made to depend on the universal instance of x in the implicant in which A occurs positively.

But, a dual variable can have more than one existential or universal instances, the number depending on the number of equivalences within whose scope it occurs.

To illustrate this, let us consider a wff

$$U_1:\ [(\forall x)\ A(x) \leftarrow\rightarrow B] \leftarrow\rightarrow C$$

The variable x is dually quantified in U_1. This, when

transformed into a wff with all equivalences paraphrased in terms of implications, will be

$$U_1': \{[((\forall x)\ A(x) \dashrightarrow B) \ \&\ (B \dashrightarrow (\forall x)\ A(x))] \dashrightarrow C\}$$
$$\&\ \{C \dashrightarrow [((\forall x)\ A(x) \dashrightarrow B) \ \&\ (B \dashrightarrow (\forall x)\ A(x))]\}$$

Thus, the single variable x has got 4 instances in U_1', 2 universal and 2 existential.

Since universal quantification distributes over $\&$, we can merge the two universal instances into a single universal variable. But we can not do so for the existential instances because existential quantification does not distribute over $\&$.

Hence, in making a wff quantifier free, we rename the two existential instances appropriately, thus creating two different existential variables. In standard normal form, U_1' would be

$$(\forall x)\ (\exists x_1)\ (\exists x_2)\ [\{ ((A(x) \dashrightarrow B) \ \&\ (B \dashrightarrow A(x_1))) \dashrightarrow C \}$$
$$\&\ \{ C \dashrightarrow ((A(x_2) \dashrightarrow B) \ \&\ (B \dashrightarrow A(x))) \}]$$

In our methodology, since we do not paraphrase equivalences in terms of implications, the above fact is transparent to us. Hence, we treat both $\exists x_1$ and $\exists x_2$ to be one and the same $\exists x$. However, as explained above, we will be choosing the polarity of C to be either positive

$U_1 + U_3 \text{---}{>}\ R_2$: $(C \dashrightarrow B) \vee (\exists x)\ {\sim}D(x)$

{ using the existential instance of x in $C \dashrightarrow [(\forall x)\ A(x) \longleftrightarrow B]$.}

So, when $(B \dashrightarrow C)$ in R_1 and $(C \dashrightarrow B)$ in R_2 are somehow eliminated, we will derive $(\exists x)\ D(x)$ and $(\exists x)\ {\sim}D(x)$, which can be resolved to give *FALSE*. But, our assertion is that such a derivation is not possible from a *satisfiable set*. The following lemmas show the soundness of the method.

Lemma 6.1

If f, g are any two Skolem functions substituted for two existential instances of a dually quantified variable in a set of wffs U, and if U is satisfiable, then two wffs $P(\ldots,f(s_1,\ldots,s_n),\ldots)$ and ${\sim}P(\ldots,g(t_1,\ldots,t_m),\ldots)$ or ${\sim}P(\ldots,f(s_1,\ldots,s_n),\ldots)$ and $P(\ldots,g(t_1,\ldots,t_m),\ldots)$ can not be deduced by NC-resolution.

Proof:

We shall prove this by *reductio ad absurdum*. Let us assume that we can derive $P(\ldots,f(s_1,\ldots,s_n),\ldots)$ and ${\sim}P(\ldots,g(t_1,\ldots,t_m),\ldots)$ from a satisfiable set of wffs U.

Since the dually quantified variable has two existential instances for which the functions f, g are substituted, it must be occurring within the scope of at least two

equivalences. The two existential instances must occur in two different implicants of the equivalence.

Let us, for example, consider such an equivalence to be of the form

$$[(Qx)\ F_1 \leftarrow\rightarrow F_2] \leftarrow\rightarrow F_3$$

where Q is some dual quantifier, and x to be the variable that is substituted by f, g.

The above wff, when all the equivalences are paraphrased in terms of implications, will be a conjunction of the following two implicants.

(1) $[\{ ((Qx)\ F_1 \rightarrow F_2) \ \&\ (F_2 \rightarrow (Qx)\ F_1) \} \rightarrow F_3]$

(2) $[F_3 \rightarrow \{ ((Qx)\ F_1 \rightarrow F_2) \ \&\ (F_2 \rightarrow (Qx)\ F_1) \}]$

For all similar instances of x, the polarity of F_1 would be same. Let us assume that f is the function symbol that is substituted for the existential instance of x in (1) and g for x in (2).

Since we can derive $P(\ldots,f(s_1,\ldots,s_n),\ldots)$, it must be only through a resolution of a wff U_1, that contains the literal $P(\ldots,y,\ldots)$, with the above wff on either F_1 or some sub-wff of F_1, such that the universal variable y gets substituted by the n-place function symbol f. Ultimately, to be able to derive a unit wff $P(\ldots,f(s_1,\ldots,s_n),\ldots)$ from this resolvent, we should eliminate some sub-wffs of U_1 and F_3, $\sim F_2$ (if Q were

universal syntactically then $\sim F_2$ else F_2) of (1) that are present in this resolvent.

Similarly, the derivation of $\sim P(\ldots,g(t_1,\ldots,t_m),\ldots)$ would involve a resolution of a wff U_2, that contains the literal $\sim P(z)$, and elimination of some sub-wffs of U_2 and $\sim F_3$, F_2 (if Q were universal syntactically then F_2 else $\sim F_2$) of (2) that are present in this resolvent.

So, the derivation of two unit wffs $P(\ldots,f(s_1,\ldots,s_n),\ldots)$, $\sim P(\ldots,g(t_1,\ldots,t_m),\ldots)$ would involve elimination of F_3, $\sim F_2$ (or F_2) and $\sim F_3$, F_2 (or $\sim F_2$). This shows the existence of two complementary sub-wffs, that are used to eliminate F_2, $\sim F_2$ or F_3, $\sim F_3$, which can be resolved upon. Then, the set U is unsatisfiable, contradicting the hypothesis.

Hence, such unit wffs $P(\ldots,f(s_1,\ldots,s_n),\ldots)$ and $\sim P(\ldots,g(t_1,\ldots,t_m),\ldots)$ can not be derived from a satisfiable set of wffs.

Similarly, we can show that a derivation of $\sim P(\ldots,f(s_1,\ldots,s_n),\ldots)$ and $P(\ldots,g(t_1,\ldots,t_m),\ldots)$ from a satisfiable set, is also not possible. □

Analogous to the above lemma, which is valid for NC-resolution, we have the following lemma for our method.

Lemma 6.2

If a wff in a set of wffs U contains a dually quantified variable x that occurs within the scope of at least two equivalences, and if U is satisfiable, then

two wffs of the form $(\exists x)\ P(x)$ and $(\exists x)\ \sim P(x)$ can not be deduced by our method.

Or, in other words, WFF-resolution together with our algorithm to obtain dependencies of an existential variable is sound.

Proof: Analogous to the proof of Lemma 6.1. □

Thus, we are justified in treating all existential instances of a dually quantified variable to be same. But, in such a case, all the existential instances are made to depend on the same universal variables, even though some are actually not dependent.

Reconsidering one of the examples we have given to demonstrate our algorithm, for the wff

$$U_1: [\{ (\forall x)\ [(\forall y)\ A \longleftrightarrow (\exists z)\ B] \} \longleftrightarrow (\forall u)\ C\ ]\ V\ \sim(\forall v)\ D$$

as input to our algorithm, it gives out

dlists: ((x y z u) (x y z u v)) ,

and sets the dependencies of existential variables as below.

$dep(x)$: ()

$dep(y)$: (x)

$dep(z)$: (x)

$dep(u)$: (x, y, z)

$dep(v)$: (x, y, z, u).

In one implicant of the sub-wff of U_1 containing equivalences, the existential instances of y and z are not dependent on x, but they are so in the other part. That is, in $[\{ (\forall x) [(\forall y) A \longleftrightarrow (\exists z) B] \} \longrightarrow (\forall u) C]$, $\exists y$ and $\exists z$ are not dependent on x, because x is also existential. However, it is guaranteed that no substitutions would be made for x in the dependency information of y or z when this implicant participates in resolution because there is no universal x in this implication for which some thing can be substituted. Hence, the dependency in such a case would essentially be dummy.

With these justifications, we proceed to look at the definitions of substitution, unification etc.

6.3 Substitution and Q-Unification

All the definitions, such as substitution, instantiation, standardization (essentially renaming of variables), and composition of substitution, of [Rob 65] carry over to our methodology, with an understanding that a variable in those definitions means universally quantified variable. For the purpose of matching or unification, an existential variable v_i with dependencies $(t_1,...,t_n)$ is treated and written as an n-place function symbol $v_i(t_1,...,t_n)$. To distinguish between a universal variable v_i and an independent existential variable v_i, the latter is written as $v_i()$.

In order to unify two sub-wffs, we need to get the sequences of terms in those sub-wffs. We can obtain the sequence of terms in a sub-wff as described below.

6.3.1 Definition (Sequence of Terms)

The *sequence of terms* in a sub-wff F is obtained as below:

(a) If $F = P(t_1,\ldots,t_n)$ where P is an n-place predicate symbol, then it is $((Q_1t_1),\ldots,(Q_nt_n))$, where Q_i is the quantification of a term t_i, taking polarity of Q_i into account.

(b) If $F = {\sim}F_1$, then it is the sequence of terms in F_1, with the notions of universal and existential quantifications of terms interchanged.

(c) If $F = F_1\ b\ F_2$ for some binary connective $b(b \neq \leftarrow\!\!\rightarrow$ if there are quantifiers in F_1 or F_2; we will explain this a little later), then it is the sequence of terms in F_1 appended to the sequence of terms in F_2; the sequence of terms obtained taking polarity into account.

(d) If $F = (Qx)\ F_1$ where Q is some quantifier, and x is some variable in F_1, then it is the sequence of terms in F_1 such that x is written as (Q_1x) where Q_1 is either Q or its complement, determined by the polarity.

As an example, the sequence of terms in

$$(\forall x)[P(x) \vee ((\forall y)\ Q(x,y) \longrightarrow (\exists z)\ \{R(x, z)\ \&\ (\forall u)\ S(z, u)\})]$$

is $((\forall x),\ (\exists y((\forall x))),\ (\exists z((\forall x))),\ (\forall u))$, which when transformed into a simple notation as described in the beginning of this section would be $(x, y(x), z(x), u)$.

It is not necessary to define sequence of terms for $F_1 \longleftrightarrow F_2$ if there are quantifiers in F_1 or F_2. If a quantifier is within the scope of $\longleftrightarrow$, we cannot resolve two sub-wffs, $(F_1 \longleftrightarrow F_2)$ and $\sim(F_1 \longleftrightarrow F_2)$ directly. Since each quantifier is dual, we cannot determine the substitutions. We will have to resolve only by parts (i.e. F_1, F_2 separately).

To convince ourselves, let us consider the following example, with two wffs whose sub-wffs contain equivalences and are complementary.

$$\begin{aligned} U_1': &\ (\forall x_1)\ [\ \{(\forall y_1)\ (\exists z_1)\ A(x_1, y_1, z_1) \longleftrightarrow \\ &\quad (\exists u_1)\ (\exists v_1)\ B(x_1, u_1, v_1)\} \vee (\exists w_1)\ C(x_1, w_1)\ ] \\ U_2': &\ (\forall x_2)\ [\ \{(\forall y_2)\ (\exists z_2)\ A(x_2, y_2, z_2) \longleftrightarrow \\ &\quad (\exists u_2)\ (\exists v_2)\ B(x_2, u_2, v_2)\} \longrightarrow (\forall w_2)\ D(x_2, w_2)\ ] \end{aligned}$$

If we can resolve on the sub-wffs which are complementary, we should get a single C and a single D with some substitutions in the resolvent. Similarly, we should get a

single C and a single D, perhaps after factoring, even when we follow a resolution sequence eliminating one literal in each step. Let us now follow a resolution sequence eliminating a single literal each time, and see whether we would be able to derive a single C and a single D in the final resolvent. We shall make these wffs quantifier free, and follow NC-resolution.

The quantifier free version of the above wffs is

U_1:

$\{[A(x_1, f_{y1}(x_1), z_{11}) \text{-->} B(x_1, f_{u1}(x_1, z_{11}), f_{v1}(x_1, z_{11}))]$
$\& [B(x_1, u_{12}, v_{12}) \text{-->} A(x_1, y_{12}, f_{z1}(x_1, z_{11}, y_{12}))]\}$
$V\ C(x_1, f_{w1}(x_1, z_{11}, y_{12}, u_{12}, v_{12}))$

U_2:

$\{ ([A(x_2, y_{22}, f_{z2}(x_2, z_{21}, y_{22})) \text{-->} B(x_2, u_{22}, v_{22})] \&$
$[B(x_2, f_{u2}(x_2, z_{21}), f_{v2}(x_2, z_{21})) \text{-->} A(x_2, f_{y2}(x_2), z_{21})])$
$\text{-->} D(x_2, w_2) \}$

The resolution sequence is

$U_1 + U_2 \text{--->} R_1$: *Resolving on* A^+ *in* U_1 *and* A^- *in* U_2 *with*
θ : $\{x_1/x_2, f_{y2}(x_2)/y_{12}, f_{z1}(x_1, z_{11}, y_{12})/z_{21}\}$ *gives*

$\{[A(x_1, f_{y1}(x_1), z_{11}) \text{-->} B(x_1, f_{u1}(x_1, z_{11}), f_{v1}(x_1, z_{11}))]$
$\& \sim B(x_1, u_{12}, v_{12})\}$
$V\ C(x_1, f_{w1}(x_1, z_{11}, f_{y2}(x_1), u_{12}, v_{12}))$
$V\ \{[A(x_1, y_{22}, f_{z2}(x_1, f_{z1}(x_1, z_{11}, f_{y2}(x_1)), y_{22}))$
$\text{-->} B(x_1, u_{22}, v_{22})]$
$\text{-->} D(x_1, w_2)\}$

R_1': Merge ~B's with θ: $\{u_{12}/u_{22}, v_{12}/v_{22}\}$

$R_1' + U_1 \longrightarrow R_2$: Resolving on B^- in R_1' and B^+ in U_1 with

θ : $\{f_{u1}(x_1, z_{11})/u_{12}, f_{v1}(x_1, z_{11})/v_{12}\}$ gives

$$
\begin{aligned}
& C(x_1, f_{w1}(x_1, z_{11}, f_{y2}(x_1), f_{u1}(x_1, z_{11}), f_{v1}(x_1, z_{11}))) \\
& V\ D(x_1, w_2) \\
& V\ \{\sim A(x_1, f_{y1}(x_1), z_{11})\ \& \\
& \qquad [B(x_1, f_{u1}(x_1, z_{11}), f_{v1}(x_1, z_{11})) \longrightarrow \\
& \qquad\qquad A(x_1, y_{12}, f_{z1}(x_1, z_{11}, y_{12}))]\} \\
& V\ C(x_1, f_{w1}(x_1, z_{11}, y_{12}, f_{u1}(x_1, z_{11}), f_{v1}(x_1, z_{11})))
\end{aligned}
$$

$R_1' + U_2 \longrightarrow R_3$: Resolving on B^- in R_1' and B^+ in U_2 with

θ : $\{x_1/x_2, f_{u2}(x_2, z_{21})/u_{12}, f_{v2}(x_2, z_{21})/v_{12}\}$ gives

$$
\begin{aligned}
& C(x_1, f_{w1}(x_1, z_{11}, f_{y2}(x_1), f_{u2}(x_1, z_{21}), f_{v2}(x_1, z_{21}))) \\
& V\ D(x_1, w_2) \\
& V\ [\{A(x_1, y_{22}, f_{z2}(x_1, z_{21}, y_{22})) \\
& \qquad \longrightarrow B(x_2, f_{u2}(x_1, z_{21}), f_{v2}(x_1, z_{21}))\} \\
& \quad \longrightarrow D(x_1, w_2)]
\end{aligned}
$$

$R_2 + R_3 \longrightarrow R_4$: Resolving on A^- in R_2 and A^+ in R_3 with

θ : $\{f_{y1}(x_1)/y_{22}, f_{z2}(x_1, z_{21}, y_{22})/z_{11}\}$ gives

$$
\begin{aligned}
& C(x_1, f_{w1}(x_1, f_{z2}(x_1, z_{21}, f_{y1}(x_1)), f_{y2}(x_1), f_{u1}(x_1, \\
& \qquad f_{z2}(x_1, z_{21}, f_{y1}(x_1))), f_{v1}(x_1, f_{z2}(x_1, z_{21}, f_{y1}(x_1)))) \\
& V\ D(x_1, w_2) \\
& V\ C(x_1, f_{w1}(x_1, f_{z2}(x_1, z_{21}, f_{y1}(x_1)), y_{12}, f_{u1}(x_1, \\
& \qquad f_{z2}(x_1, z_{21}, f_{y1}(x_1))), f_{v1}(x_1, f_{z2}(x_1, z_{21}, f_{y1}(x_1))))) \\
& V\ C(x_1, f_{w1}(x_1, z_{11}, f_{y2}(x_1), f_{u2}(x_1, z_{21}), f_{v2}(x_1, z_{21}))) \\
& V\ D(x_1, w_2)\ V\ D(x_1, w_2)
\end{aligned}
$$

In R_4, we can merge all D's into a single D, but we cannot do so for C's. Hence, we can not resolve on two sub-wffs containing <--> if quantifiers are present within the scope of <-->.

So from the polarity of the wff being resolved we can determine the sequence of terms because each quantifier will then have a single polarity. For example, in a wff $[(\forall x)\ A$ <--> $B]$, if we decide to take the polarity of $(\forall x)\ A$ to be negative, then x is only existentially quantified.

Two sub-wffs are *unifiable*, if the sequence of terms in one, and the sequence of terms in the S-equivalent form of the other are unifiable. Since the terms to be unified contain quantifiers, we call our unification *Q-unification*. However, the basic unification algorithm of [Rob 65] extends to our methodology.

For example, consider the above wffs U_1' and U_2'. The sequences of terms in positive $A(x_1, y_1, z_1)$ in U_1' and negative $A(x_2, y_2, z_2)$ in U_2' respectively are $((\forall x_1), (\forall y_1), (\exists z_1(x_1, y_1)))$ and $((\forall x_2), (\exists y_2(x_2)), (\forall z_2))$. If written in the simple notation that we have described earlier, they are $(x_1, y_1, z_1(x_1, y_1))$ and $(x_2, y_2(x_2), z_2)$. Unification algorithm results in the mgu $\{x_1/x_2, y_2(x_1)/y_1, z_1(x_1, y_2(x_1))/z_2\}$.

For unifying two existential variables, we check for their origin and if it is the same, then only we proceed to unify their dependencies, just as in the case of unifying

two Skolem functions. Since the existential variables do not change in syntactic form during resolution, by *same origin of existential variables* we mean *their syntactic equivalence*. For instance, the terms $x(y, z)$ and $x(u(), v)$ unify with $\{u()/y, v/z\}$, where as the terms $x(y, z)$ and $w(u(), v)$ do not because x, w are not syntactically same.

6.4 WFF-resolution

This is basically NC-resolution, but with quantifiers in place. We define a WFF-resolvent as follows.

6.4.1 Definition (WFF-resolvent)

For any wffs S_1, S_2, F and G, if F occurs positively in S_1 $(S_1\langle F\rangle)$ and G occurs negatively in S_2 $(S_2\langle G\rangle)$, and θ is the mgu of F and G under Q-unification, such that $F\theta = G\theta = H$, then the result of simplifying

$$S_1\theta\{FALSE/H\} \vee S_2\theta\{TRUE/H\}$$

is the *WFF-resolvent* of S_1 and S_2.

The simplification may involve dropping, adding or merging quantifiers. For example, let us consider the wffs

U_1: $(\forall x)\ (A(x) \rightarrow B(x))$
U_2: $(\exists y())\ (C(y) \rightarrow A(y))$

in which the atom with A is complementary. We can resolve these two wffs on A with the mgu $\{y()/x\}$ and the resolvent is

$$R_1: (\exists y())\ [B(y) \vee \sim C(y)].$$

Thus, we drop the quantifiers $(\forall x)$, $(\exists y())$ respectively from U_1, U_2 and add a quantifier $(\exists y())$ as prefix to the resolvent because the existential variable y substitutes for the universal variable x.

When we *instantiate* a universal variable (i.e. substitute an existential variable for a universal variable), we drop the universal quantification of that variable and add existential quantification of the variable being substituted in the resolvent. Such dropping and addition respectively of universal and existential quantifiers can be viewed analogous to the *Universal Instantiation* and *Existential Generalization* in *Quantification Theory* [Qui 61]; of course, not exactly the same, for the method of logical deduction set forth in [Qui 61] is natural deduction, a derivative of Gentzen [Gen 34] and Jaskowski's [Jas 34].

As for *renaming* variables in a resolvent, we do so only for universally quantified variables, but not for any existentially quantified variable. Essentially, an existential variable *retains* its syntactic form obtained from the origin wff of the variable through out the refutation.

Let us now look at the application of WFF-resolution considering a simple example from [Qui 61]. The English version of the problem is

Premises:

The guard searched all who entered the premises except those who were accompanied by members of the firm,

Some of Fiorecchio's men entered the premises unaccompanied by anyone else,

The guard searched none of the Fiorecchio's men;

Conclusion:

Some of Fiorecchio's men were members of the firm.

We define the following predicates, in order to represent the above problem in First-Order Logic.

P(x) : *x is a person that entered the premises,*

G(x) : *x was searched by the guard,*

H(x, y): *x was accompanied by y,*

M(y) : *y was a member of the firm,*

FM(x) : *x was one of Fiorecchio's men.*

The first-order logic translation of the above problem is

(1) $(\forall x)\ [(P(x)\ \&\ {\sim}G(x)) \longrightarrow (\exists y)\ (H(x, y)\ \&\ M(y))]$

(2) $(\exists x)\ [FM(x)\ \&\ P(x)\ \&\ (\forall y)\ (H(x, y) \longrightarrow FM(y))]$

(3) $(\forall x)\ (FM(x) \longrightarrow {\sim}G(x))$

Conclusion:

$(4)\ (\exists x)\ (FM(x)\ \&\ M(x))$

Renaming the variables properly and adding functional dependencies of existential variables, the set with negated conclusion is

U_1:

$(\forall x_1)\ [(P(x_1)\ \&\ {\sim}G(x_1))\ \text{-->}\ (\exists y_1(x_1))\ (H(x_1,\ y_1)\ \&\ M(y_1))]$

U_2:

$(\exists x_2())\ [FM(x_2)\ \&\ P(x_2)\ \&\ (\forall y_2)\ (H(x_2,\ y_2)\ \text{-->}\ FM(y_2))]$

U_3: $(\forall x_3)\ [FM(x_3)\ \text{-->}\ {\sim}G(x_3)]$

U_4: ${\sim}[(\exists x_4)\ (FM(x_4)\ \&\ M(x_4))]$

The resolution sequence that leads to *FALSE* is

$U_1 + U_2$ ---> R_1: *Resolving on* P^- *in* U_1 *and* P^+ *in* U_2 *with*

θ: $\{(\exists x_2())/(\forall x_1)\}$ *gives*

$(\exists x_2())\ [{\sim}G(x_2)\ \text{-->}\ (\exists y_1(x_2))\ (H(x_2,\ y_1)\ \&\ M(y_1))]$

$R_1 + U_3$ ---> R_2: *Resolving on* G^+ *in* R_1 *and* G^- *in* U_3 *with*

θ: $\{(\exists x_2())/(\forall x_3)\}$ *gives*

$(\exists x_2())\ \{\ [(\exists y_1(x_2))\ (H(x_2,\ y_1)\ \&\ M(y_1))]\ V\ {\sim}FM(x_2)\}$

$R_2 + U_2$ ---> R_3: *Resolving on* FM^- *in* R_2 *and* FM^+ *in* U_2 *with*

θ: $\{\}$ *gives*

$(\exists x_2())\ [\ (\exists y_1(x_2))\ (H(x_2,\ y_1)\ \&\ M(y_1))\ ]$

$R_3 + U_4$ ---> R_4: *Resolving on* M^+ *in* R_3 *and* M^- *in* U_4 *with*

θ: $\{(\exists x_2())/(\forall x_4)\}$ *gives*

$(\exists x_2())\ [\sim FM(x_2)]$

$R_4 + U_2$ ---> R_5: *Resolving on* FM^- *in* R_4 *and* FM^+ *in* U_2 *with*

θ: $\{\}$ *gives*

FALSE.

Thus, if *FALSE* is reached as a resolvent, then the set of wffs U is unsatisfiable.

Some more examples are given in Chapter 10. We shall deal with the soundness and completeness of WFF-resolution in the next section. .

6.5 Soundness and Completeness of WFF-resolution

We shall prove these properties of WFF-resolution by showing that it is equivalent to quantifier free NC-resolution, which has been proved to be sound and complete [Mur 82].

For these proofs, we shall induce on the total number of equivalences in the set of wffs U taking the base step to be when there are no equivalences. In regard to this base step, we will prove a few results before going to the main proof.

Lemma 6.3

If U is a set of wffs without equivalences, and QF is its quantifier free version, then for every wff U_i from U there exists an equivalent wff QF_j in QF

following dependencies for U.

$dep(x)$: ()

$dep(y)$: (x)

$dep(z)$: $(x\ y)$

where as the quantifier free form of U would have different functional dependencies substituted for existential variables as given below.

$$\{[A(a) \dashrightarrow B(y)] \,\&\, [B(f(x, y)) \dashrightarrow A(x)]\} \vee C(g(x, y))$$

Thus, the existential instance of the variable y substituted by $f(x, y)$ has functional dependencies that are different from those obtained by our algorithm. However, we will prove in Theorem 6.2 that WFF-resolution is equivalent to NC-resolution.

Lemma 6.4

If two wffs U_1, U_2 can be resolved to give R_1 by WFF-resolution and two quantifier free wffs QF_1, QF_2 can also be resolved to give QFR_1, and $U_1 \equiv QF_1$, $U_2 \equiv QF_2$, then the resolvents, if the resolutions are performed on similar sub-wffs, satisfy $R_1 \equiv QFR_1$. (The wffs are assumed to be free of equivalences).

Proof:

To show that $R_1 \equiv QFR_1$, it suffices to show that the corresponding unifications result in equivalent

substitutions. If the substitutions are same for every two terms t_1, t_2, then the composite substitution or the mgu must be same in both resolutions.

In unifying t_1, t_2, we have four different possibilities. We denote WFF-resolution by *WR* and quantifier free resolution by *QFR*.

Case (i): When both t_1, t_2 are variables in *QFR* and are universally quantified in *WR*, both unifications result in trivially equivalent substitutions $\{t_1/t_2\}$ or $\{t_2/t_1\}$.

Case (ii): When t_1 is a constant or a function and t_2 is a variable in *QFR*, and t_1 is either existentially quantified or a constant or a function and t_2 is universally quantified in *WR*, then also both the unifications result in the same substitution $\{t_1/t_2\}$.

Case (iii): When t_1 is a variable and t_2 is a constant or a function in *QFR*, and t_1 is universally quantified and t_2 is either existentially quantified or a constant or a function in *WR*, then it is a dual case of Case (ii).

Case (iv): When both t_1, t_2 are constants or functions in *QFR*, and are either existentially quantified or constants or functions in *WR*.

If they are constants, then the normal unification will be successful only if they are syntactically same. If they are functions, in addition to the syntactic equivalence, the sub-terms of the functions should unify. So, essentially we

are checking for the *origin* of these constants or functions in the set of wffs. Similarly, we check for the syntactic equivalence of existential variables in *Q-unification* and unify the sub-terms if the existential variables are dependent.

Thus, if the terms are constants, then the substitutions in both cases are empty. If terms are functions and dependent variables, then the substitutions in both cases are those that result from unifying the sub-terms. The equivalence of such substitutions can again be explained in the same lines as this proof.

Thus, in all possible cases of unification, the substitutions are shown to be *equivalent*, and hence the lemma.

□

To illustrate how the above lemma is true, we reconsider the example that we have given in Chapter 2, but with quantifiers in place.

The unsatisfiable set of wffs, with functional dependencies substituted, is

S_1:

$(\forall x_1)\ [(E(x_1)\ \&\ \sim V(x_1)) \dashrightarrow (\exists y_1(x_1))\ (S(x_1, y_1)\ \&\ C(y_1))]$

S_2: $(\exists x_2())\ [P(x_2)\ \&\ E(x_2)\ \&\ (\forall y_2)\ (S(x_2, y_2) \dashrightarrow P(y_2))]$

S_3: $(\forall x_3)\ [P(x_3) \dashrightarrow \sim V(x_3)]$

S_4: $\sim(\exists x_4)\ [P(x_4)\ \&\ C(x_4)]$

We shall follow the same resolution sequence as given for its quantifier free version in Chapter 2.

The wffs S_2, S_3 can be resolved on positive $P(x_2)$ in S_2 and negative $P(x_3)$ in S_3 with the substitution $\{(\exists x_2())/(\forall x_3)\}$ which is equivalent to the substitution $\{a/x\}$ done in NC-resolution (Chapter 2) to give an equivalent resolvent

$$S_2 + S_3 \dashrightarrow R_1: (\exists x_2()) \sim V(x_2) \quad ;\{(\exists x_2())/(\forall x_3)\}$$

{ some independent individual constant x_2 is not a VIP }

Thus, with a notion that the existential variable x_2 and Skolem constant a are equivalent and the existential variable y_1 and Skolem function f are also equivalent, the following resolution steps are also equivalent to their quantifier free analogues.

$$S_1 + R_1 \dashrightarrow R_2:$$

$$(\exists x_2())\ [E(x_2) \dashrightarrow (\exists y_1(x_2))\ (S(x_2, y_1)\ \&\ C(y_1))]$$
$$;\{(\exists x_2())/(\forall x_1)\}$$

{ If x_2 has entered the country, then he is searched by the custom official $y_1(x_2)$ }

$S_2 + R_2 \longrightarrow R_3$:

$(\exists x_2()) \ [(\exists y_1(x_2)) \ (S(x_2, y_1) \ \& \ C(y_1))] \quad ;\{\}$

{ established that the custom official $y_1(x_2)$ searched x_2}

$S_2 + R_3 \longrightarrow R_4$:

$(\exists x_2()) \ (\exists y_1(x_2)) \ [P(x_2) \ \& \ E(x_2) \ \& \ P(y_1)] \quad ;\{(\exists y_1(x_2))/(\forall y_2)\}$

{ established that $y_1(x_2)$ is also a drug pusher }

$S_4 + R_4 \longrightarrow R_5$:

$(\exists x_2()) \ (\exists y_1(x_2)) \ \sim C(y_1) \quad ;\{(\exists y_1(x_2))/(\forall x_4)\}$

{ established that $y_1(x_2)$ is not a custom official }

This contradicts the previously established assertion R_3 that $y_1(x_2)$ is a custom official, thus refuting the given set.

$R_3 + R_5 \longrightarrow R_6$: *FALSE* $;\{\}$

Based on these results, we shall now formally prove that WFF-resolution *(WR)* and quantifier free NC-resolution *(QFR)* are equivalent when there are no equivalences in the given set of wffs on which these resolutions are performed.

Theorem 6.1

If U is an unsatisfiable set of wffs without equivalences, and QF is its quantifier free version, then *FALSE* can be deduced from U by *WR* iff *FALSE* can be deduced from QF by *QFR*.

Proof:

To prove the theorem both ways, it suffices to show that the resolution sequences with equivalent wffs in WR and QFR are equivalent.

Let $QFR_1,\ldots,QFR_n$ be the sequence of resolution steps in QFR and the resolvent in QFR_i be QR_i. Let $WR_1,\ldots,WR_k$ be the resolution sequence in WR and the resolvent in WR_i be R_i.

By Lemma 6.3, we have, for every wff in U there is a corresponding and equivalent wff in QF and *vice-versa*. If any two wffs QF_i, QF_j are resolved in QFR_1, then the equivalent resolution step in WR, say WR_1, must consist of wffs U_i, U_j, such that $QF_i \equiv U_i$ and $QF_j \equiv U_j$, and resolution must be on equivalent sub-wffs. Then, by Lemma 6.4, the resolvents QR_1 and R_1 satisfy $QR_1 \equiv R_1$. Hence, the derivation of an equivalent wff must be equally long in both QFR and WR. Hence, $k = n$.

We shall prove the equivalence of the two resolution sequences by inducing on the number of resolution steps. We show that every QFR_i can be transformed into a valid, equivalent WR_i.

Base step: First resolution steps QFR_1, WR_1.

If wffs QF_i, QF_j are resolved in QFR_1, then these wffs must be from the initial set QF. By Lemma 6.3, we can choose two wffs U_i, U_j from U, such that $QF_i \equiv U_i$ and $QF_j \equiv U_j$, for resolution step WR_1. Then, by Lemma 6.4,

the resolvents QR_1, R_1 satisfy $QR_1 \equiv R_1$. Thus, we have an equivalent resolution step in WR for the one in QFR.

Hypothesis: For resolution steps $< m$, we have, for every resolution step in QFR, an equivalent step in WR.

Inductive step: Consider the m^{th} resolution step QFR_m. We have 3 different cases here.

Case (i): When both the wffs in QFR_m are from QF.

As explained for base step, we can find an equivalent WR_m for QFR_m.

Case (ii): When one wff is from QF and the other from the set of resolvents $\{QR_1,\ldots,QR_{m-1}\}$.

From Lemma 6.3, we can choose an equivalent wff U_i for the one from QF. From the hypothesis, we can always choose a wff from the resolvent set of WR, $\{R_1,\ldots,R_{m-1}\}$, such that it is equivalent to the one from $\{QR_1,\ldots,QR_{m-1}\}$. By Lemma 6.4, the resolvent of these two wffs in WR is equivalent to QR_m. Hence, we have an equivalent WR_m for QFR_m.

Case (iii): When both the wffs are from the resolvent set of QFR, $\{QR_1,\ldots,QR_{m-1}\}$.

From the hypothesis, we can always find two wffs in $\{R_1,\ldots,R_{m-1}\}$, such that they are equivalent to the wffs in QFR_m. Hence, by Lemma 6.4, we have a resolvent R_m in WR, such that $R_m \equiv QR_m$. Thus, we have an equivalent WR_m for QFR_m.

Hence, in all possible cases, we have shown that there exists a WR_m, which is equivalent to QFR_m.

Now, by induction, for every resolution sequence in QFR, we have an equivalent sequence of resolutions in WR.

Hence, $FALSE$ can be deduced from U in WR iff $FALSE$ can be deduced from QF in QFR. □

We shall now look at the proof to show that WFF-resolution is sound and complete, when equivalences are present.

Theorem 6.2

If U is an unsatisfiable set of wffs, and QF is its quantifier free version, then $FALSE$ can be deduced by WR iff $FALSE$ can be deduced by QFR.

Proof:

To prove the theorem both ways, it suffices to show the equivalence of WR and QFR.

We prove this by inducing on n, the total number of equivalences in the set of wffs U.

If $n = 0$, then we have by Theorem 6.1, that WR and QFR are equivalent.

Suppose that the Theorem is true for $n < m$. Consider, now, a set of wffs with m equivalences. Let us assume that a wff U_1 contains an equivalence. If we show that there exists an equivalent (equivalence of functional dependencies

also) wff in QF for this U_1, then the proof for the equivalence of WR and QFR goes in the same lines as in the proof of Theorem 6.1.

Now, we have 3 different cases here.

Case (i): When there are no quantifiers within the scope of this equivalence, this equivalence does not affect the equivalence of WR and QFR. Hence, we are through.

Case (ii): When there is only one quantifier within the scope of equivalence, then also we are through, because we can always assume that the implicant containing the existential instance of the variable, precedes the other one.

Case (iii): When there is more than one quantifier within the scope of equivalence.

Let a sub-wff containing this equivalence have the form

$$(Q_{11}x_1)\ldots(Q_{1k}x_k)\ F_1 \longleftrightarrow (Q_{21}y_1)\ldots(Q_{21}y_l)\ F_2$$

For any existential variable that follows this sub-wff containing equivalence, we have exactly $k+l$ universal variables, k from the first sub-wff of this equivalence and l from the second, coming out of this sub-wff as dependency, because every variable is both universally quantified and existentially quantified. This would be the case in QF also. Hence, for any functional dependency of an existential variable that follows this equivalence, there is no difference between U and QF.

This will have the following two implicants:

$$(1)\quad (Q_{11}x_1)\ldots(Q_{1k}x_k)\ F_1 \dashrightarrow (Q_{21}y_1)\ldots(Q_{2l}y_l)\ F_2$$

$$(2)\quad (Q_{21}y_1)\ldots(Q_{2l}y_l)\ F_2 \dashrightarrow (Q_{11}x_1)\ldots(Q_{1k}x_k)\ F_1$$

The functional dependencies, generated from one implicant, for an existential variable in the other part, are being ignored by our algorithm of section 6.3, which otherwise would have been taken into account in the set *QF*. But, as explained in section 6.2, we use only one implicant of any equivalence in a resolution. So, irrespective of what substitutions could have been made in the functional dependencies of an existential variable in one part, when the other part is reduced to *FALSE*, the whole sub-wff gets reduced to *FALSE*, making such existential variables disappear. Hence, such dependencies need not be considered.

Hence, depending on the chosen polarity of a sub-wff in this equivalence, we will be using only one of the dual quantifications of a variable quantified as $(Q_{ij}v_j)$ in a resolution step. We can always assume that the implicant that participates in resolution precedes the other part.

The resolvent in *WR* when one implicant is eliminated completely, will be equivalent to the resolvent in *QFR* when the corresponding implicant is also eliminated completely.

Thus, we have shown that there exists an equivalent wff

in *QF* for U_i, and also that the descendants of both these wffs are also equivalent, though not after a single resolution step, but after some resolution steps in which the implicants participating in resolution have been erased completely.

Thus, by induction, we have for every sub-wff containing an equivalence in a wff U_i of U, there is an equivalent wff in *QF* containing equivalent sub-wff in terms of implications.

Now, going along the same lines as in the proof of Theorem 6.1, we will have the equivalence of *WR* and *QFR*. By this and also by Lemma 6.2 which establishes soundness of *WR*, we have, *FALSE* can be deduced in *WR* from a set *U* iff *FALSE* can be deduced in *QFR* from *QF*, the quantifier free version of *U*.

Thus, WFF-resolution has been proved to be sound and complete. □

Chapter 7

MANY-SORTED RESOLUTION FOR WELL-FORMED FORMULAE

In this chapter we introduce many-sortedness into WFF-resolution.

7.1 Introduction

This chapter extends the methodology of converting unsorted set of wffs into its many-sorted version to include wffs with quantifiers in place. This chapter also describes how to perform WFF-resolution on the MS set. MS WFF-resolution is proved to be sound and complete.

7.2 Many-sortedness for wffs

All the definitions of Chapter 3 carry over to the present method with an understanding that if a term is a variable, it is universally quantified and if it is a constant or a function, it can be either an existentially quantified variable or a constant or a function.

Given a set of wffs (need not be quantifier free) U = $\{U_1, U_2, ...\}$, we interpret it as a conjunction of its members. So any wff in U of the form F_1 & F_2 can be replaced by the wffs F_1, F_2. We apply the algorithm, described in section 6.2, to obtain the dependencies of

existential variables in each wff of U and then use the method, described in the next section to convert it into a many-sorted set.

7.2.1 Unsorted to Many-sorted

The definition of s.form and other details can be looked up in section 3.3.1.

All the terms are initially assumed to be of sort T, and updated as the sorting process continues. All the rules given in Chapter 3 can be restated for wffs with quantifiers in place as below. In the following rules, $[Qx]$ where Q is some quantifier, represents an *optional quantifier* in the place indicated; it may be present somewhere in the wff U, having the sub-wff F within its scope.

Rule 1:

Remove all wffs of the form $(\forall v)\ (U' \longrightarrow U'')$ where U' and U'' are s.forms of a variable v. If S' and S'' are the sort-symbols got from U' and U'' respectively, then $S' \subseteq S''$ is a sort axiom.

Rule 2:

If a wff U has a sub-wff F,

(a) which is positive in it, and

(b) has all occurrences of a universally quantified variable v, and

(c) is of the form $[\forall v]\ (F_1 \longrightarrow F_2)$ or $(\exists v)\ F_1 \longrightarrow F_2$, where F_1 is the s.form of v, and F_2

is some wff,

then F_1 can be removed, and the sort of v in U is given by the sort corresponding to F_1.

Rule 3:

If a wff U has a sub-wff F,

(a) which is positive in it, and

(b) has all occurrences of a constant or a function symbol or an existentially quantified variable v, and

(c) is of the form $[\exists v]\ (F_1\ \&\ F_2)$ or $(\exists v)\ F_1\ \&\ F_2$, where F_1 is an s.form of v, and F_2 is some wff or $TRUE$,

then F_1 can be removed, and the sort of v in U is given by the sort corresponding to F_1.

Rule 4:

If a wff U has a sub-wff F,

(a) which is positive in it, and

(b) has all occurrences of a constant or a function symbol or an existentially quantified variable v, and

(c) is of the form $[\exists v]\ (F_1 \dashrightarrow F_2)$ or $[\forall v]\ F_1 \dashrightarrow F_2$, where F_1 is the s.form of v and v has been found to be of the sort corresponding to F_1, and F_2 is some wff,

then F_1 can be removed.

Since we deal here with dual quantifiers and variables, as we do not paraphrase equivalences in terms of implications, we present two additional rules, by which we can obtain sort information for dual variables. Addition of these rules does not affect soundness and completeness, for they are only a composition of rules 2 and 3.

Rule 5:

If a wff U has a sub-wff F,

(a) which has all occurrences of a variable v, and

(b) is of the form $(\forall v)\ {\sim}F_1 \leftarrow\!\!\rightarrow F_2$, where F_1 is the s.form of v, and F_2 is some wff,

then F can be replaced by ${\sim}F_2$, and the sort of v in U is given by the sort corresponding to F_1.

Rule 6:

If a wff U has a sub-wff F,

(a) which has all occurrences of a variable v, and

(b) is of the form $(\exists v)\ F_1 \leftarrow\!\!\rightarrow F_2$, where F_1 is the s.form of v, and F_2 is some wff,

then F can be replaced by F_2, and the sort of v in U is given by the sort corresponding to F_1.

The conditions to be satisfied for a unary predicate to become a sort symbol are the same as given in Chapter 3. The algorithm for converting an unsorted set of wffs U into

its many-sorted version M, is as described in section 3.3.1, with the rule set having the above 6 rules.

We simplify the wffs in the process of applying the above rules; the simplification may involve *dropping quantifiers*, such as in $(\forall v)\ (F_1 \rightarrow F_2)$ when F_1 is removed and F_2 is free of v.

7.3 Resolution and Unification

The *Many-sorted WFF-resolution (MWFF-resolution)* is the WFF-resolution of Chapter 6 with MS unification.

7.3.1 Definition (MWFF-resolvent)

If P is a sub-wff that occurs positively in the wff F_1 and Q occurs negatively in F_2, and θ is the MS mgu under Q-unification of P and Q, then simplifying

$$F_1\theta\{FALSE/P\theta\} \vee F_2\theta\{TRUE/Q\theta\}$$

will give the *MWFF-resolvent* of F_1 and F_2.

The *MS mgu* is the sort compatible mgu as defined in section 3.4.

Since many-sorted resolution is not complete [Wal 82] without the additional rule, namely the *Weakening rule*, MWFF-resolution also requires such a rule.

7.3.2 Weakening rule

If W is a wff with a universally quantified variable v in it, and t is a term, such that $[t] \subseteq [v]$, then

the wff U with t substituted for v, (i.e. $U\theta$, $\theta = \{t/v\}$) is called the *weakened variant* of U.

We shall now consider the same example that we have given in Chapter 6 to illustrate the conversion rules as well as the MWFF-resolution.

The unsorted set of wffs, with functional dependencies obtained by our algorithm (section 6.2) and substituted in wffs, is

U_1:
$(\forall x_1)\ [(E(x_1)\ \&\ {\sim}V(x_1)) \dashrightarrow (\exists y_1(x_1))\ (S(x_1, y_1)\ \&\ C(y_1))]$

U_2: $(\exists x_2())\ [P(x_2)\ \&\ E(x_2)\ \&\ (\forall y_2)\ (S(x_2, y_2) \dashrightarrow P(y_2))]$

U_3: $(\forall x_3)\ [P(x_3) \dashrightarrow {\sim}V(x_3)]$

U_4: ${\sim}(\exists x_4)\ [P(x_4)\ \&\ C(x_4)]$

The many-sorted version of the above set would be

Sort axioms: None.

Sorts of terms: $\forall x_1 : E$, $\exists y_1(x_1) : C$,
$\exists x_2() : E$, $\forall y_2 : T$,
$\forall x_3 : T$, $\forall x_4 : C$.

MS sentences:

M_1: $(\forall x_1)\ [{\sim}V(x_1) \dashrightarrow (\exists y_1(x_1))\ S(x_1, y_1)]$

M_2: $(\exists x_2())\ [P(x_2)\ \&\ (\forall y_2)\ (S(x_2, y_2) \dashrightarrow P(y_2))]$

M_3: $(\forall x_3)\ [P(x_3) \dashrightarrow {\sim}V(x_3)]$

M_4: ${\sim}(\exists x_4)\ P(x_4)$

The resolution sequence would be essentially analogous to its quantifier free version given in Chapter 3.

Here, M_1 and M_2 can be resolved as the substitution $\{(\exists x_2())/(\forall x_1),\ (\exists y_1(x_2))/(\forall y_2)\}$ is sort compatible and makes the atoms with S identical.

$M_1 + M_2 \longrightarrow R_1$:

$(\exists x_2())\ (\exists y_1(x_2))\ [\sim(\sim V(x_2))\ V\ (P(x_2)\ \&\ P(y_1))]$

$;\{(\exists x_2())/(\forall x_1),\ (\exists y_1(x_2))/(\forall y_2)\}$

And further resolutions would be

$M_4 + R_1 \longrightarrow R_2$: $(\exists x_2())\ \sim(\sim V(x_2))$ $;\{(\exists y_1(x_2))/(\forall x_4)\}$

$M_3 + R_2 \longrightarrow R_3$: $(\exists x_2())\ \sim P(x_2)$ $;\{(\exists x_2())/(\forall x_3)\}$

$M_2 + R_3 \longrightarrow R_4$: $FALSE$ $;\{\}$

Some more examples are given in Chapter 10. We shall now look at how the soundness and completeness properties of WFF-resolution are preserved in the case of MWFF-resolution.

7.4 Soundness and Completeness of MWFF-resolution

Based on the soundness and completeness results of WFF-resolution (Chapter 6), we will now prove that the MWFF-resolution together with the Weakening rule is also sound and complete.

The soundness and completeness proofs for MNC-resolution [Jay 87] carry over to the MWFF-resolution, with

the understanding that

(i) a term is as defined in Chapter 6, and

(ii) a variable is a universally quantified variable, and

(iii) U is an unsorted set of wffs with quantifiers in place, and

(iv) M is its many-sorted version, and

(v) UR denotes an inference step in unsorted WFF-resolution, and

(vi) MR denotes a many-sorted inference step; either MWFF-resolution or a weakening rule application.

Hence, we will just state the results here.

The unsorted set of wffs U, with quantifiers in place, is considered to be the union of two sets of wffs U_a and U_b as in [Jay 87], where

$U_a = \{ U_{ai} \mid U_{ai}$ is a wff in U, with at least one atom whose predicate is not a sort in MS $\}$

$U_b = \{ U_{bi} \mid U_{bi}$ is a wff in U, with atoms all of whose predicates are sorts in MS $\}$

7.4.1 Soundness

Lemma 7.1

For any two sorts P and Q, if $P \subseteq Q$, then the wff $(\forall x)\ (P(x) \rightarrow Q(x))$ for some variable x, was either present in US or could have been deduced.

Lemma 7.2

If P is a sort in MS, then the wff $P(c)$ where c is a constant or a function, or the wff $(\exists x)\ P(x)$ where x is a variable, is either present or can be deduced.

Lemma 7.3

For a sort P and a constant or a function c, the wff $P(c)$, or for a variable x, $(\exists x)\ P(x)$ can be deduced in US iff $[c] \subseteq P$ or $[x] \subseteq P$ in MS.

Theorem 7.1 (Soundness Theorem)

If M is the MS translation of the set of wffs U with quantifiers in place, then

$$M \overset{MR^*}{\vdash} FALSE \dashrightarrow U \overset{UR^*}{\vdash} FALSE$$

7.4.2 Completeness

Lemma 7.4

If P is a sort in MS, then the wff $(\forall x)\ P(x)$ where x is a variable, can neither exist nor be deduced in US.

Lemma 7.5

If P is a sort in MS, then the wff $(\forall x)\ {\sim}P(x)$ where x is a variable, can neither exist nor be deduced in US.

Lemma 7.6

The set of wffs U_b is satisfiable.

Theorem 7.2 (Completeness Theorem)

If M is the MS translation of the set of wffs U with quantifiers in place, then

$$U \overset{UR^*}{\longmapsto} FALSE \dashrightarrow M \overset{MR^*}{\longmapsto} FALSE$$

Thus, MWFF-resolution is also a sound and complete proof procedure.

Chapter 8

EQUALITY-BASED RESOLUTION FOR WELL-FORMED FORMULAE

The importance of equality in reasoning is well known [MaW 86] and [DiH 86]. We have seen the disadvantages of explicit use of the equality axioms in resolution, in Chapter 4, while extending RUE resolution to the non-clausal case. We have also seen the need for retaining form and structure of wffs in resolution, in Chapter 6. Here we combine the two and discuss a refutation procedure for wffs with quantifiers in place and equality.

8.1 Introduction

All the definitions of Chapter 4 extend to the present case, with the notion of terms, substitution, unification etc., as in Chapter 6. So, all the terms which are constants or functions in Chapter 4 can now be either constants or functions or existential variables, and all variables are universal variables. The dependencies can be obtained by the algorithm in Chapter 6.

This chapter describes how to perform NCRUE resolution with quantifiers in place, and shows that the soundness and

8.3 Soundness and Completeness of WRUE resolution

These properties of WRUE resolution can be established by showing its equivalence to NCRUE resolution similar to how these properties of WFF-resolution were proved based on those of NC-resolution.

All proofs for WFF-resolution (Chapter 6) would serve as the required proofs for the soundness and completeness of WRUE resolution, with WFF-resolution replaced by WRUE resolution and NC-resolution by NCRUE resolution. Hence, we will just state the results here.

Since Lemma 6.3 is about the set of wffs U and its quantifier free version QF, it remains same for the present case. Lemma 6.4 gets modified as stated below.

Lemma 8.1

If two wffs U_1, U_2 can be resolved to give R_1 by WRUE resolution and two quantifier free wffs QF_1, QF_2 can also be resolved to give QFR_1, and $U_1 \equiv QF_1$, $U_2 \equiv QF_2$, then the resolvents, if the resolutions are performed on similar sub-wffs using similar disagreement sets and similar unification rules, satisfy $R_1 \equiv QFR_1$. (The wffs are assumed to be free of equivalences).

Proof:

In addition to the arguments given in the proof for Lemma 6.4, we have the following.

Whenever one of the two rules of inference, say NCNRF, is applied in NCRUE resolution, we assume that an analogous rule would be applied in WRUE resolution.

When none of the two terms t_1, t_2 is a universal variable and the unification fails, we add the equality $t_1 = t_2$ as a condition into the resolvent, in both WRUE and NCRUE resolutions. Thus, the resolvents would be equivalent in all cases. □

Theorem 8.1

If *U* is an E-unsatisfiable set of wffs without *<-->'s*, and *QF* is its quantifier free version, then *FALSE* can be deduced from *U* by WRUE resolution iff *FALSE* can be deduced from *QF* by NCRUE resolution.

Theorem 8.2

If *U* is an E-unsatisfiable set of wffs, and *QF* is its quantifier free version, then *FALSE* can be deduced from *U* by WRUE resolution iff *FALSE* can be deduced from *QF* by NCRUE resolution.

The proofs for both the theorems are exactly along the lines of the corresponding theorems in Chapter 6 with the definitions changed as mentioned above.

Thus, WRUE resolution is also a sound and complete proof procedure.

Chapter 9

MANY-SORTED AND EQUALITY-BASED WFF-RESOLUTION

There is a need for a single proof procedure that captures all the three aspects that we have mentioned in the introductory chapter. We have developed, in the previous chapters, proof procedures that capture any two of the three aspects. Now, it remains for everything to be integrated into a single proof procedure. We do that in this chapter.

9.1 Introduction

The general sorting rules have already been extended to the case of wffs with quantifiers in place, in Chapter 7. This chapter extends the sorting rules for equality to wffs with quantifiers in place. This chapter also describes how to perform many-sorted WRUE resolution on the MS set of wffs, and shows that such a proof procedure is sound and complete.

9.2 Many-sortedness

All the definitions of Chapter 7 carry over to the present case.

Any set of wffs U (need not be quantifier free) can be interpreted as a conjunction of its members. Hence, any

wff of the form F_1 & F_2 can be replaced by the wffs F_1, F_2. We follow the following procedure to make a given set of wffs U ready for refutation.

Step 1:

Apply the algorithm of section 6.2 to each wff in U to obtain dependencies of existential variables.

Step 2:

Apply the algorithm of section 3.3.1 with the general sorting rules of section 7.2.1 to convert U into its many-sorted version M.

Step 3:

Apply the sorting rules for equality as described in section 9.2.1 until no more application is possible, to strengthen the sortedness of M.

9.2.1 Sorting rules for Equality

Following are the sorting rules for equality, of section 5.2, modified to be applicable to wffs with quantifiers in place.

Rule EQ_1:

If there is a positive unit equality, $[\exists c]$ $[Qt]$ $(c = t)$, where Q is some quantifier and c is a variable in which case it is existentially quantified, or a constant or a function symbol of sort s_c and t is a term of sort s_t, then the sort of c is changed into $s_c \cap s_t$.

Rule EQ_2:

If there is a positive unit equality, $[\forall x] [Qt] (x = t)$, where Q is some quantifier and x is a universally quantified variable of sort S_x and t is a term of sort S_t, then a sort axiom $S_x \subseteq S_t$ is added.

We use the symmetry property of equality when necessary while applying these rules.

The conditions to be satisfied by a unary predicate to become a sort symbol are the same as in Chapter 3. The simplification as explained in section 7.2.1 may be done while applying these rules.

We shall now look at how to perform WRUE resolution on the MS set thus obtained.

9.3 Many-sorted WRUE resolution

The *Many-sorted WRUE resolution (MWRUE resolution)* is the WRUE resolution of Chapter 8 with MS unification. The substitutions made in WRUE resolution should be sort compatible.

All the definitions of Chapter 8 or those that have been extended to it also extend to the present case, with the substitution or unification being MS substitution or MS unification. Hence, we omit the definitions of the rules of inference here. One can refer to the definitions of Chapter 5 for inference rules, with all the notions of WFF-resolution. See Chapter 10 for examples.

9.4 Soundness and Completeness of MWRUE resolution

The proofs go along the same lines as those for MNCRUE resolution (Chapter 5), with NCRUE resolution replaced by WRUE resolution and MNCRUE resolution by MWRUE resolution and the understanding that

(i) a term is as defined in Chapter 6, and

(ii) a variable is a universally quantified variable, and

(iii) U is an unsorted set of wffs with quantifiers in place, and

(iv) M is its many-sorted version, and

(v) UR denotes an inference step in unsorted WRUE resolution, and

(vi) MR denotes a many-sorted inference step, either MWRUE resolution or a weakening rule application.

Hence, we will just state the results here.

The unsorted set of wffs U, with quantifiers in place, is considered to be the union of two sets of wffs U_a and U_b as in [Jay 87], where

$U_a = \{ U_{ai} \mid U_{ai}$ is a wff in U, with at least one atom whose predicate is not a sort in MS $\}$

$U_b = \{ U_{bi} \mid U_{bi}$ is a wff in U, with atoms all of whose predicates are sorts in MS $\}$

9.4.1 Soundness

Lemma 9.1

For any two sorts P and Q, if $P \subseteq Q$, then one of the following is true.

(a) the wff $(\forall x)\ (P(x) \dashrightarrow Q(x))$ for some variable x, was either present in US or could have been deduced.

(b) the positive unit equality, $(\forall x)\ [Qt]\ (x = t)$, where Q is some quantifier, for some variable x and some term t of sorts P and Q respectively, was either present or could have been deduced.

Lemma 9.2

If P is a sort in MS, then the wff $P(c)$ where c is a constant or a function, or the wff $(\exists x)\ P(x)$ where x is a variable, is either present or could have been deduced.

Lemma 9.3

For a sort P and a constant or a function or an existentially quantified variable v, $[v] \subseteq P$ iff one of the following is true.

(a) the wff $[\exists v]\ P(v)$ can be deduced in US.

(b) the wff $[\exists v]\ P(v)$ or $[\exists u]\ P(u)$ where u is another constant or function or existential variable, such that $[\exists v]\ [\exists u]\ (v = u)$ is a positive equality in US, can be deduced in US.

Theorem 9.1 (Soundness Theorem)

If M is the MS translation of the set of wffs U with quantifiers in place, then

$$M \overset{MR^*}{\longmapsto} FALSE \dashrightarrow U \overset{UR^*}{\longmapsto} FALSE$$

Proof: Along the same lines as the proof for Theorem 5.1. □

9.4.2 Completeness

Lemma 9.4

If P is a sort in MS, then the wff $(\forall x)\ P(x)$ where x is a variable, can neither exist nor be deduced in US.

Lemma 9.5

If P is a sort in MS, then the wff $(\forall x)\ {\sim}P(x)$ where x is a variable, can neither exist nor be deduced in US.

Lemma 9.6

The set of wffs U_b is E-satisfiable.

Theorem 9.2 (Completeness Theorem)

If M is the MS translation of the set of wffs U with quantifiers in place, then

$$U \overset{UR^*}{\longmapsto} FALSE \dashrightarrow M \overset{MR^*}{\longmapsto} FALSE$$

Proof: Along the same lines as the proof for Theorem 5.2.

□

Thus MWRUE resolution is also proved to be sound and complete.

EXAMPLES AND APPLICATIONS

In this chapter we present some examples to illustrate the application of various proof procedures that we have introduced in this thesis.

10.1 NCRUE resolution

We shall consider an example from *Group Theory* to illustrate NCRUE resolution.

Group Theory axioms:

U_1:	$f(e, x) = x$	*Left identity*
U_2:	$f(x, e) = x$	*Right identity*
U_3:	$f(g(x), x) = e$	*Left inverse*
U_4:	$f(x, g(x)) = e$	*Right inverse*
U_5:	$f(f(x, y), z) = f(x, f(y, z))$	*Associativity*

Theorem:

$$(\forall x)\ (\forall y)\ [f(y, x) = e \longrightarrow f(x, y) = e]$$

The quantifier free version of the negated theorem would be

U_6: ~[f(b, a) = e --> f(a, b) = e].

Here we can resolve U_6, U_3 with an mgpu {b/x} and a disagreement set {a;g(b)} to give the resolvent

U_6 + U_3 ---> R_1: ~[a = g(b)]

And further resolutions that lead to *FALSE* are as follows:

U_2 + R_1 ---> R_2: ~[f(g(b), e) = a] ;{g(b)/x}
U_5 + R_2 ---> R_3:
~[f(f(g(b), y), z) = a & e = f(y, z)] ;{g(b)/x}
U_1 + R_3 ---> R_4:
[(f(g(b), y) = e) --> ~(e = f(y, a))] ;{a/x, a/z}
U_6 + R_4 ---> R_5: ~[f(g(b), b) = e] ;{b/y}
U_3 + R_5 ---> R_6: FALSE ;{b/x}

We shall now consider a problem of *Elementary Geometry.* Alfred Tarski [Tar 51] uses only two predicates to represent the axioms of geometry in FOL. B(x, y, z) represents *"y is between x and z"* and $L(x_1, y_1, x_2, y_2)$ represents *"the distance from x_1 to y_1 is the same as the distance from x_2 to y_2".* The following are the axioms of

Tarski, but without his full axiom of continuity, which is not easily represented in FOL.

Identity axiom for betweenness:

U_1: $(\forall x)\ (\forall y)\ [B(x, y, x) \longrightarrow x = y]$

Transitivity axiom for betweenness:

U_2:

$(\forall x)(\forall y)(\forall z)(\forall u)\ [(B(x, y, u)\ \&\ B(y, z, u)) \longrightarrow B(x, y, z)]$

Connectivity axiom for betweenness:

U_3:

$$(\forall x)\ (\forall y)\ (\forall z)\ (\forall u)\ [\{B(x, y, z)\ \&\ B(x, y, u)\ \&\ \sim(x = y)\} \longrightarrow \{B(x, z, u) \vee B(x, u, z)\}]$$

Reflexivity axiom for equidistance:

U_4: $(\forall x)\ (\forall y)\ L(x, y, y, x)$

Identity axiom for equidistance:

U_5: $(\forall x)\ (\forall y)\ (\forall z)\ [L(x, y, z, z) \longrightarrow x = y]$

Transitivity axiom for equidistance:

$$U_6\text{: } (\forall x)\ (\forall y)\ (\forall z)\ (\forall u)\ (\forall v)\ (\forall w)\ [L(x, y, z, u)\ \&\ L(x, y, v, w) \longrightarrow L(z, u, v, w)]$$

Pasch's axiom:

U_7:

$$(\forall t)\ (\forall x)\ (\forall y)\ (\forall z)\ (\forall u)\ [\{B(x, t, u)\ \&\ B(y, u, z)\} \longrightarrow (\exists w)\ \{B(x, w, y)\ \&\ B(z, t, w)\}]$$

Euclid's axiom:

U_8:

$$(\forall x)\ (\forall y)\ (\forall z)\ (\forall t)\ (\forall u)\ [\{B(x, u, t)\ \&\ B(y, u, z)\ \&\ \sim(x = u)\} \longrightarrow (\exists v)\ (\exists w)\ \{B(x, z, w)\ \&\ B(x, y, v)\ \&\ B(w, t, v)\}]$$

Five-segment axiom:

U_9: $(\forall x_1)\ (\forall x_2)\ (\forall y_1)\ (\forall y_2)\ (\forall z_1)\ (\forall z_2)\ (\forall u_1)\ (\forall u_2)$

$[\{L(x_1, y_1, x_2, y_2)\ \&\ L(y_1, z_1, y_2, z_2)\ \&\ L(x_1, u_1, x_2, u_2)$

$\&\ L(y_1, u_1, y_2, u_2)\ \&\ B(x_1, y_1, z_1)\ \&\ B(x_2, y_2, z_2)$

$\&\ \sim(x_1 = y_1)\ \&\ \sim(y_1 = z_1)\} \dashrightarrow L(z_1, u_1, z_2, u_2)]$

Axiom of segment construction:

U_{10}: $(\forall x)(\forall y)(\forall u)(\forall v)(\exists z)\ [B(x, y, z)\ \&\ L(y, z, u, v)]$

Lower dimension axiom:

U_{11}:

$(\exists x)\ (\exists y)\ (\exists z)\ [\sim B(x, y, z)\ \&\ \sim B(y, z, x)\ \&\ \sim B(z, x, y)]$

Upper dimension axiom:

U_{12}: $(\forall x)\ (\forall y)\ (\forall z)\ (\forall u)\ (\forall v)$

$[\{L(x, u, x, v)\ \&\ L(y, u, y, v)\ \&\ L(z, u, z, v)\ \&\ \sim(u = v)\}$

$\dashrightarrow \{B(x, y, z)\ V\ B(y, z, x)\ V\ B(z, x, y)\}]$

Weakened continuity axiom:

U_{13}: $(\forall x_1)\ (\forall x_2)\ (\forall y_1)\ (\forall z_1)\ (\forall z_2)\ (\forall u_1)$

$[\{L(u_1, x_1, u_1, x_2)\ \&\ L(u_1, z_1, u_1, z_2)\ \&\ B(u_1, x_1, z_1)\ \&$

$B(x_1, y_1, z_1)\} \dashrightarrow (\exists v_1)\ \{L(u_1, y_1, u_1, v_1)\ \&\ B(x_2, v_1, z_2)\}]$

The clausal set for the above problem consists of 55 clauses, 20 from the above axioms and 35 from equality axioms for the above problem [WOL 84].

Let us consider a theorem,

For all points x and y, y is between x and y.

i.e. $(\forall x)\ (\forall y)\ B(x, y, y)$.

The quantifier free form of the above set would be

$$U_1\colon\ [B(x, y, x) \longrightarrow x = y]$$

$$U_2\colon\ [(B(x, y, u) \,\&\, B(y, z, u)) \longrightarrow B(x, y, z)]$$

$$U_3\colon\ [\{B(x, y, z) \,\&\, B(x, y, u) \,\&\, \sim(x = y)\} \longrightarrow \{B(x, z, u) \vee B(x, u, z)\}]$$

$$U_4\colon\ L(x, y, y, x)$$

$$U_5\colon\ [L(x, y, z, z) \longrightarrow x = y]$$

$$U_6\colon\ [L(x, y, z, u) \,\&\, L(x, y, v, w) \longrightarrow L(z, u, v, w)]$$

$$U_7\colon\ [\{B(x, t, u) \,\&\, B(y, u, z)\} \longrightarrow \{B(x, f_1(t, x, y, z, u), y) \,\&\, B(z, t, f_1(t, x, y, z, u))\}]$$

$$U_8\colon\ [\{B(x, u, t) \,\&\, B(y, u, z) \,\&\, \sim(x = u)\} \longrightarrow \{B(x, z, f_2(t, x, y, z, u)) \,\&\, B(x, y, f_3(t, x, y, z, u)) \,\&\, B(f_2(t, x, y, z, u), t, f_3(t, x, y, z, u))\}]$$

$$U_9\colon\ [\{L(x_1, y_1, x_2, y_2) \,\&\, L(y_1, z_1, y_2, z_2) \,\&\, L(x_1, u_1, x_2, u_2) \,\&\, L(y_1, u_1, y_2, u_2) \,\&\, B(x_1, y_1, z_1) \,\&\, B(x_2, y_2, z_2) \,\&\, \sim(x_1 = y_1) \,\&\, \sim(y_1 = z_1)\} \longrightarrow L(z_1, u_1, z_2, u_2)]$$

$$U_{10}\colon\ [B(x, y, f_4(x, y, u, v)) \,\&\, L(y, f_4(x, y, u, v), u, v)]$$

$$U_{11}\colon\ [\sim B(c_1, c_2, c_3) \,\&\, \sim B(c_2, c_3, c_1) \,\&\, \sim B(c_3, c_1, c_2)]$$

$$U_{12}\colon\ [\{L(x, u, x, v) \,\&\, L(y, u, y, v) \,\&\, L(z, u, z, v) \,\&\, \sim(u = v)\} \longrightarrow \{B(x, y, z) \vee B(y, z, x) \vee B(z, x, y)\}]$$

U_{13}:

$[\{L(u_1, x_1, u_1, x_2) \,\&\, L(u_1, z_1, u_1, z_2) \,\&\, B(u_1, x_1, z_1) \,\&$
$B(x_1, y_1, z_1)\} \longrightarrow \{L(u_1, y_1, u_1, f_5(x_1, y_1, z_1, x_2, z_2, u_1))$
$\& \, B(x_2, f_5(x_1, y_1, z_1, x_2, z_2, u_1), z_2)\}]$

The quantifier free negated theorem is

U_{14}: $\sim B(a, b, b)$

The clausal solution with the explicit use of equality axioms requires 10 steps where as the solution by NCRUE resolution requires only 3 steps as given below.

$U_{14} + U_{10} \longrightarrow R_1$: $\sim(f_4(a, b, u, v) = b)$;$\{a/x, b/y\}$
$U_5 + R_1 \longrightarrow R_2$:
$\sim L(b, f_4(a, b, u, v), z, z)$;$\{b/x_1, f_4(a, b, u, v)/y_1\}$
$U_{10} + R_2 \longrightarrow R_3$:
FALSE ;$\{b/y_1, a/x_1, z/u_1, z/v_1, z/u, z/v\}$

10.2 MNCRUE resolution

We shall solve here the *Agatha's problem* [Pel 86] which can be stated as below:

Premises:

Some one who lives in Dreadsbury Mansion killed Aunt Agatha.

Agatha, the butler, and Charles live in Dreadsbury

Mansion, and are the only people who live therein.

A killer always hates his victim, and is never richer than his victim.

Charles hates no one that Aunt Agatha hates.

Agatha hates everyone except the butler.

The butler hates everyone not richer than Aunt Agatha.

The butler hates everyone Agatha hates.

No one hates everyone.

Agatha is not the butler.

Theorem:

Agatha killed herself.

We define the predicates as given below:

$L(x)$: x *lives in Dreadsbury Mansion*,

$K(x, y)$: x *killed* y,

$R(x, y)$: x *is richer than* y,

$H(x, y)$: x *hates* y

The constants a, b and c represent respectively *Agatha, the butler* and *Charles*.

With the above predicates, the FOL translation of the problem is

U_1 : $(\exists x)\ [L(x)\ \&\ K(x, a)]$

U_2 : $[L(a) \& L(b) \& L(c)]$

U_3 : $(\forall x)\ [L(x) \longrightarrow x = a \lor x = b \lor x = c]$

U_4 : $(\forall x)\ (\forall y)\ [K(x, y) \longrightarrow H(x, y)]$

U_5 : $(\forall x)\ (\forall y)\ [K(x, y) \longrightarrow \sim R(x, y)]$

U_6 : $(\forall x)\ [H(a, x) \longrightarrow \sim H(c, x)]$

U_7 : $(\forall x)\ [x \neq b \longrightarrow H(a, x)]$

U_8 : $(\forall x)\ [\sim R(x, a) \longrightarrow H(b, x)]$

U_9 : $(\forall x)\ [H(a, x) \longrightarrow H(b, x)]$

U_{10}: $(\forall x)\ (\exists y)\ \sim H(x, y)$

U_{11}: $a \neq b$

U_{12}: $\sim K(a, a)$

The quantifier free version of the above set of wffs would be

U_1 : $[L(d) \& K(d, a)]$

U_2 : $[L(a) \& L(b) \& L(c)]$

U_3 : $[L(x) \longrightarrow x = a \lor x = b \lor x = c]$

U_4 : $[K(x, y) \longrightarrow H(x, y)]$

U_5 : $[K(x, y) \longrightarrow \sim R(x, y)]$

U_6 : $[H(a, x) \longrightarrow \sim H(c, x)]$

U_7 : $[x \neq b \longrightarrow H(a, x)]$

U_8 : $[\sim R(x, a) \longrightarrow H(b, x)]$

U_9 : $[H(a, x) \longrightarrow H(b, x)]$

U_{10}: $\sim H(x, f(x))$

U_{11}: $a \neq b$

U_{12}: $\sim K(a, a)$

The many-sorted set of wffs for the above problem consists of

Sort axioms: None.

Sorts of terms: $x{:}L$, $a{:}L$, $b{:}L$,

$c{:}L$, $d{:}L$, $y{:}T$,

$z{:}T$, $f(y){:}T$.

MS sentences:

M_1 : $K(d, a)$

M_2 : $[x = a \lor x = b \lor x = c]$

M_3 : $[K(y, z) \longrightarrow H(y, z)]$

M_4 : $[K(y, z) \longrightarrow {\sim}R(y, z)]$

M_5 : $[H(a, y) \longrightarrow {\sim}H(c, y)]$

M_6 : $[y \neq b \longrightarrow H(a, y)]$

M_7 : $[{\sim}R(y, a) \longrightarrow H(b, y)]$

M_8 : $[H(a, y) \longrightarrow H(b, y)]$

M_9 : ${\sim}H(y, f(y))$

M_{10}: $a \neq b$

M_{11}: ${\sim}K(a, a)$

Here we can resolve M_1, M_{11} to a negative equality, because the disagreement $d{:}a$ is viable under sorting as there is a positive equality in M_2 such that d MS unifies with x.

$M_1 + M_{11} \longrightarrow R_1$: ${\sim}(d = a)$;{}

And further resolutions which refute the above set are

$M_6 + M_{10} \dashrightarrow R_2 : H(a, a)$;{a/y}

$M_5 + R_2 \dashrightarrow R_3 : \sim H(c, a)$;{a/y}

$M_3 + R_3 \dashrightarrow R_4 : \sim K(c, a)$;{c/y, a/z}

$M_1 + R_4 \dashrightarrow R_5 : \sim(d = c)$;{}

$M_2 + R_5 \dashrightarrow R_6 : [d = a \vee d = b]$;{d/x}

$R_1 + R_6 \dashrightarrow R_7 : d = b$;{}

$M_4 + M_7 \dashrightarrow R_8 : \sim K(y, a) \vee H(b, y)$;$\{y/y_4, a/z_4\}$

{with the variables y, z in M_4 renamed to y_4, z_4 respectively.}

$M_1 + R_8 \dashrightarrow R_9 : H(b, d)$;{d/y}

$M_9 + R_9 \dashrightarrow R_{10} : \sim(f(b) = d)$;{b/y}

$M_6 + R_{10} \dashrightarrow R_{11} : [b = d \dashrightarrow H(a, f(b))]$;{f(b)/y}

$M_8 + R_{11} \dashrightarrow R_{12} : \sim(b = d) \vee H(b, f(b))$;{f(b)/y}

$M_9 + R_{12} \dashrightarrow R_{13} : \sim(b = d)$;{b/y}

$R_7 + R_{13} \dashrightarrow R_{14} : FALSE$;{}

10.3 WFF-resolution

We shall consider again a small problem to illustrate WFF-resolution.

The problem consists of only one wff,

$$U_1 : \sim[(\forall x)(\forall y)\ (\exists z)(\forall w)\ \{(P(x)\ \&\ Q(y)) \dashrightarrow (R(z)\ \&\ S(w))\} \dashrightarrow \{(\exists u)\ (\exists v)\ (P(u)\ \&\ Q(v)) \dashrightarrow (\exists t)\ R(t)\}]$$

which is unsatisfiable.

The above wff with the functional dependencies obtained by our algorithm substituted is

U_1:

$$\sim[(\forall x)(\forall y)(\exists z(x, y))(\forall w)\ \{(P(x)\ \&\ Q(y)) \longrightarrow (R(z)\ \&\ S(w))\} \longrightarrow \{(\exists u(x, y, w))(\exists v(x, y, w))\ (P(u)\ \&\ Q(v)) \longrightarrow (\exists t)\ R(t)\}]$$

Here we can observe that the sub-wffs $(P(x)\ \&\ Q(y))$ and $(P(u)\ \&\ Q(v))$ are complementary and hence we can resolve U_1 with itself with the substitution $\{(\exists u(x, y, w))/(\forall x'), (\exists v(x, y, w))/(\forall y')\}$, when all the universal variables v_i in one wff are renamed to v_i', to give the following resolvent:

$$U_1 + U_1 \longrightarrow R_1\text{:}\ (\exists u(x, y, w))\ (\exists v(x, y, w)) \sim[(\exists z(u(x, y, w), v(x, y, w)))\ (\forall w')\ (R(z)\ \&\ S(w')) \longrightarrow \{(\exists u(x', y', w'))\ (\exists v(x', y', w'))\ (P(u')\ \&\ Q(v')) \longrightarrow (\exists t')\ R(t')\}]$$

Now we can resolve R_1 with U_1 on sub-wffs $R(z)$ and $R(t)$ with the substitution $\{(\exists z(u(x, y, w), v(x, y, w)))/(\forall t)\}$ to give a resolvent which simplifies to *FALSE* thus refuting the set.

$U_1 + R_1 \longrightarrow R_2$: *FALSE*.

We shall now consider a problem from [Cha 79] reproduced in [Pel 86] concerning set equality.

Set equality (Q) is defined as having exactly the same members. Prove set equality is symmetric. Predicate F stands for *"is an element of"*. The wffs are

Definition of Set equality:

$$U_1: (\forall x)\ (\forall y)\ [Q(x, y) \longleftrightarrow (\forall z)\ (F(z, x) \longleftrightarrow F(z, y))]$$

Theorem:

$$U_2: (\forall u)\ (\forall v)\ (Q(u, v) \longleftrightarrow Q(v, u))$$

The wffs with negated theorem and with functional dependencies substituted are

U_1:

$$(\forall x)\ (\forall y)\ [Q(x, y) \longleftrightarrow (\forall z(x, y))\ (F(z, x) \longleftrightarrow F(z, y))]$$

$$U_2: \sim(\forall u())\ (\forall v())\ [Q(u, v) \longleftrightarrow Q(v, u)]$$

Since the variable z is dual it will be dependent on x, y only when it acts as existential variable.

The refutation sequence and the substitutions therein are as given below:

$U_1 + U_2 \longrightarrow R_1$:

(∃u()) (∃v()) [~(∀z(u, v)) (F(z, u) <--> F(z, v) V ~Q(v, u)]

;{(∃u())/(∀x), (∃v())/(∀y)}

$U_1 + R_1 \longrightarrow R_2$:

(∃u()) (∃v()) [~(∀z(u, v)) (F(z, u) <--> F(z, v)) V

~(∀z(v, u)) (F(z, v) <--> F(z, u))]

;{(∃v())/(∀x), (∃u())/(∀y)}

$U_1 + R_2 \longrightarrow R_3$:

(∃u()) (∃v()) [~Q(u, v) V ~(∀z(v, u)) (F(z, v) <--> F(z, u))]

;{(∃u())/(∀x), (∃v())/(∀y), (∃z(u, v))/(∀z_1)}

$U_1 + R_3 \longrightarrow R_4$: (∃u()) (∃v()) ~Q(u, v)

;{(∃u())/(∀x), (∃v())/(∀y), (∃z(v, u))/(∀z_1)}

and merging into ~Q(u, v).

$U_2 + R_4 \longrightarrow R_5$: ~[(∀u()) (∀v()) ~Q(v, u)] ;{}

$U_1 + R_5 \longrightarrow R_6$: (∃u()) (∃v()) (∀z) (F(z, v) <--> F(z, u))

;{(∃v())/(∀x), (∃u())/(∀y)}

$R_2 + R_6 \longrightarrow R_7$:

(∃u()) (∃v()) [~(∀z(u, v)) (F(z, u) <--> F(z, v))]

;{(∃z(v, u))/(∀z_1)}

$R_6 + R_7 \longrightarrow R_8$: FALSE ;{(∃z(u, v))/(∀$z_1$)}

10.4 MWFF-resolution

Let us consider a problem from [Pel 86] which consists of the following wffs.

U_1: $(\exists x)\ P(x) \longleftrightarrow (\exists y)\ Q(y)$

U_2: $(\forall x_1)\ (\forall y_1)\ [(P(x_1)\ \&\ Q(y_1)) \longrightarrow (R(x_1) \longleftrightarrow S(y_1))]$

U_3: $[(\forall x_2)\ (P(x_2) \longrightarrow R(x_2)) \longleftrightarrow (\forall y_2)\ (Q(y_2) \longrightarrow S(y_2))]$

The wffs get modified as given below when the theorem is negated and the functional dependencies are substituted.

U_1: $(\exists x())\ P(x) \longleftrightarrow (\exists y(x))\ Q(y)$

U_2: $(\forall x_1)\ (\forall y_1)\ [(P(x_1)\ \&\ Q(y_1)) \longrightarrow (R(x_1) \longleftrightarrow S(y_1))]$

U_3:

$\sim[(\forall x_2())\ (P(x_2) \longrightarrow R(x_2)) \longleftrightarrow (\forall y_2(x_2))\ (Q(y_2) \longrightarrow S(y_2))]$

Using the rules of section 7.2.1, we obtain the many-sorted version of the above set as

Sort axioms: None.

Sorts of terms:

$\exists x()$	$:P,$	$\exists y(x)$	$:Q,$
$\forall x$	$:P,$	$\forall y$	$:Q,$
$\forall x_1$	$:P,$	$\forall y_1$	$:Q,$
$\exists x_2()$	$:P,$	$\exists y_2(x_2)$	$:Q,$
$\forall x_2$	$:P,$	$\forall y_2$	$:Q.$

MS sentences:

M_1: $(\forall x_1)\ (\forall y_1)\ [R(x_1) \longrightarrow S(y_1)]$

M_2: $\sim[(\forall x_2())\ R(x_2) \longleftrightarrow (\forall y_2(x_2))\ S(y_2)]$

We can resolve M_1 and M_2 on sub-wffs $R(x_1)$ and $R(x_2)$ using the sort compatible substitution $\{(\forall x_2)/(\forall x_1)\}$

to give

$$M_1 + M_2 \text{ ---> } R_1\text{: } (\forall y_1)\ S(y_1)\ V\ \sim[\sim(\forall y_2(x_2))\ S(y_2)]$$

Merging into $S(y_1)$ we get

$$(\forall y_1)\ S(y_1)$$

While merging, we dropped the dependencies (x_2) for y_2, because y_2 is no more dual but only universal. Further resolutions and the substitutions that refute the set are as follows:

$$M_2 + R_1 \text{ ---> } R_2\text{: } \sim[\sim(\forall x_2())\ R(x_2)]\quad ;\{(\exists y_2(x_2))/(\forall y_1)\}$$

$$M_2 + R_2 \text{ ---> } R_3\text{: } \sim[(\forall y_2(x_2))\ S(y_2)]\ ;\{(\exists x_2())/(\forall x_2')\}$$

{ with x_2 in R_2 renamed to x_2' }

$$R_1 + R_3 \text{ ---> } R_4\text{: } FALSE\quad ;\{(\exists y_2(x_2))/(\forall y_1)\}$$

10.5 WRUE resolution

We shall consider the *Elementary Geometry* problem that we have given to illustrate NCRUE resolution. We will give here in quantifier-in-place form only those wffs that have participated in resolution.

$$U_5\ : (\forall x)\ (\forall y)\ (\forall z)\ [L(x,\ y,\ z,\ z) \text{ --> } x = y]$$

$$U_{10}\text{: } (\forall x)(\forall y)\ (\forall u)(\forall v)\ (\exists z)\ [B(x,\ y,\ z)\ \&\ L(y,\ z,\ u,\ v)]$$

U_{14}: $\sim(\forall x)\ (\forall y)\ B(x, y, y)$.

These wffs with the functional dependencies substituted would be

U_5 : $(\forall x_1)\ (\forall y_1)\ (\forall z_1)\ [L(x_1, y_1, z_1, z_1) \longrightarrow x_1 = y_1]$

U_{10}: $(\forall x_2)\ (\forall y_2)\ (\forall u)\ (\forall v)\ (\exists z_2(x_2, y_2, u, v))$

$[B(x_2, y_2, z_2)\ \&\ L(y_2, z_2, u, v)]$

U_{14}: $\sim(\forall x_3())\ (\forall y_3())\ B(x_3, y_3, y_3)$.

The resolutions that lead to *FALSE* are as given below:

$U_{14} + U_{10} \longrightarrow R_1$:

$(\exists x_3())\ (\exists y_3())\ (\exists z_2(x_3, y_3, u, v))\ \sim(z_2 = y_3)$

$;\{(\exists x_3())/(\forall x_2), (\exists y_3())/(\forall y_2)\}$

$U_5 + R_1 \longrightarrow R_2$:

$(\exists x_3())\ (\exists y_3())\ (\exists z_2(x_3, y_3, u, v))\ (\forall z_1)\ \sim L(y_3, z_2, z_1, z_1)$

$;\{(\exists y_3())/x_1, (\exists z_2(x_3, y_3, u, v))/y_1\}$

$U_{10} + R_2 \longrightarrow R_3$: FALSE

$;\{(\exists y_3())/(\forall y_2), (\exists x_3())/(\forall x_2), (\forall z_1)/(\forall u_1), (\forall z_1)/(\forall v_1),$

$(\forall z_1)/(\forall u), (\forall z_1)/(\forall v)\}$

10.6 MWRUE resolution

We shall consider again the *Agatha's problem* for illustration. The many-sorted set of wffs with quantifiers in place would be

Sort axioms: None.

Sorts of terms: $\forall x_2 : L$, $a : L$, $b : L$,

$c : L$, $\exists x_1() : L$, $\forall x_i : T$, $i=3,9$

$\forall y_3 : T$, $\forall y_4 : T$, $\exists y_9(x_9) : T$.

MS sentences:

M_1 : $(\exists x_1())\ K(x_1, a)$

M_2 : $(\forall x_2)\ [x_2 = a \vee x_2 = b \vee x_2 = c]$

M_3 : $(\forall x_3)\ (\forall y_3)\ [K(x_3, y_3) \longrightarrow H(x_3, y_3)]$

M_4 : $(\forall x_4)\ (\forall y_4)\ [K(x_4, y_4) \longrightarrow \sim R(x_4, y_4)]$

M_5 : $(\forall x_5)\ [H(a, x_5) \longrightarrow \sim H(c, x_5)]$

M_6 : $(\forall x_6)\ [x_6 \neq b \longrightarrow H(a, x_6)]$

M_7 : $(\forall x_7)\ [\sim R(x_7, a) \longrightarrow H(b, x_7)]$

M_8 : $(\forall x_8)\ [H(a, x_8) \longrightarrow H(b, x_8)]$

M_9 : $(\forall x_9)\ (\exists y_9(x_9))\ \sim H(x_9, y_9)$

M_{10}: $a \neq b$

M_{11}: $\sim K(a, a)$

The resolutions corresponding to those we have given in section 10.2 are as follows:

$M_1 + M_{11} \longrightarrow R_1$: $(\exists x_1())\ \sim(x_1 = a)$;$\{\}$

$M_6 + M_{10} \longrightarrow R_2$: $H(a, a)$;$\{a/(\forall x_6)\}$

$M_5 + R_2 \longrightarrow R_3$: $\sim H(c, a)$;$\{a/(\forall x_5)\}$

$M_3 + R_3 \longrightarrow R_4$: $\sim K(c, a)$;$\{c/(\forall x_3), a/(\forall y_3)\}$

$M_1 + R_4 \longrightarrow R_5$: $(\exists x_1())\ \sim(x_1 = c)$;$\{\}$

$M_2 + R_5 \longrightarrow R_6$: $(\exists x_1())\ [x_1 = a \vee x_1 = b]$

;$\{(\exists x_1())/(\forall x_2)\}$

$R_1 + R_6 \;\longrightarrow\; R_7 : (\exists x_1()) \; (x_1 = b) \qquad ;\{\}$

$M_4 + M_7 \;\longrightarrow\; R_8 : (\forall x_7) \; [\sim K(x_7, a) \vee H(b, x_7)]$

$;\{(\forall x_7)/(\forall x_4), \; a/(\forall y_4)\}$

$M_1 + R_8 \;\longrightarrow\; R_9 : (\exists x_1()) \; H(b, x_1) \qquad ;\{(\exists x_1())/(\forall x_7)\}$

$M_9 + R_9 \;\longrightarrow\; R_{10}: (\exists y_9(b)) \; (\exists x_1()) \sim(y_9 = x_1) \; ;\{b/(\forall x_9)\}$

$M_6 + R_{10} \;\longrightarrow\; R_{11}: (\exists y_9(b)) \; (\exists x_1()) \; [b = x_1 \longrightarrow H(a, y_9)]$

$;\{(\exists y_9(b))/(\forall x_6)\}$

$M_8 + R_{11} \;\longrightarrow\; R_{12}:$

$(\exists x_1()) \; (\exists y_9(b)) \; [\sim(b = x_1) \vee H(b, y_9))]$

$;\{(\exists y_9(b))/(\forall x_8)\}$

$M_9 + R_{12} \;\longrightarrow\; R_{13}: (\exists x_1()) \sim(b = x_1) \qquad ;\{b/(\forall x_9)\}$

$R_7 + R_{13} \;\longrightarrow\; R_{14}: FALSE \qquad ;\{\}$

Chapter 11

CONCLUSIONS

We have discussed three necessary improvements to make *Automated Reasoning* tractable in the introductory chapter. To state them again,

(i) Retain the original form of wffs in resolution.

(ii) Introduce domain information into rules of inference (e.g. Sorting).

(iii) Include special relations (e.g. equality) in reasoning in an efficient way.

RUE resolution [DiH 86] has been developed only for clause sets to handle equality in an efficient way. The advantages of NC-resolution are well known [Mur 82], [MaW 80]. Hence, we have extended RUE resolution to operate on quantifier free wffs. We have proved that NCRUE resolution in open form is sound and complete. We have also extended criteria like viability, RUE unification and Equality restriction, of RUE resolution in strong form to the non-clausal case and shown that the completeness is preserved independently with viability and equality restriction. The completeness of RUE resolution in strong form has only been conjectured. The same conjecture holds for the strong form

of NCRUE resolution that incorporates all the criteria together. Thus we have a non-clausal deduction system to handle equality in an efficient way.

The advantages of many-sorted logic are well established [Hen 72], [Hay 71]. It has been extended to clausal resolution and paramodulation [Wal 82] and subsequently to non-clausal resolution [Jay 87]. Since we are aiming at theorem provers with the three features mentioned earlier, we have extended many-sorted logic to NCRUE resolution to provide a single proof procedure with all the three features, except that the wffs are modified to be quantifier free. We have added an additional rule to the original rule set for finding sorts [Jay 87], by which more sorts may be extracted, without violating the soundness and completeness of MNC-resolution. We have also added two more rules for sorting based on equality, analogous to those given by Schmidt [Sch 85] for clause sets, and shown that many-sorted NCRUE resolution together with the weakening rule [Wal 82] and the method for automatic generation of sort information is sound and complete.

Keeping the first feature in mind, we have developed WFF-resolution, NC-resolution for wffs with quantifiers in place, because NC-resolution requires wffs to be quantifier free and not all sentences (especially those with quantifiers in the scope of an equivalence) can be made quantifier free, without paraphrasing some of its connectives in terms of the others. We have given an algorithm to obtain depen-

dencies of existential variables in a wff and shown that WFF-resolution together with this algorithm is sound and complete. Thus, except for maintaining dependencies along with the existential variables, analogous to the functional arguments in Skolem functions in a wff we do not modify the wff in any way. Thus, the first feature is incorporated into a refutation proof procedure so that the intuitive idea behind using various connectives and quantifiers is preserved.

Since WFF-resolution is one step ahead of NC-resolution in terms of retaining the form of wffs in resolution, we have extended WFF-resolution to many-sorted logic and given the required changes for the sorting rules to include quantifiers in place. We have also given two additional sorting rules by which we can obtain the sort information for dual variables, without paraphrasing $\longleftrightarrow$'s in terms of $\longrightarrow$'s. We have shown that the soundness and completeness of WFF-resolution is preserved in many-sorted logic, together with the weakening rule and the method for automatic generation of sorts, analogous to those properties of NC-resolution in many-sorted logic. Thus we have a single proof procedure incorporating the first two features.

We have given a NCRUE deduction system for wffs with quantifiers in place, named WRUE, and shown that its soundness and completeness essentially follows along the same lines as that of WFF-resolution. Thus we have a proof procedure that can handle equality in an efficient way as well

as retain the form of the wffs.

Finally, we have integrated WFF-resolution, many-sorted resolution and equality-based resolution into a single proof procedure, MWRUE resolution, by extending WRUE resolution to include sortedness. We have given the sorting rules for equality with quantifiers in place. We have shown that the MWRUE resolution together with the weakening rule, and the mechanisms for obtaining dependencies and generating sort information automatically is sound and complete.

Thus, we have developed a single proof procedure that captures all the three features described earlier, required to make *Automated Reasoning* more tractable.

The present work can be extended in a number of ways as indicated below.

Many-sorted calculus can be extended to include negated sorts. But, it has been observed that a simple modification in Unification algorithm, where a term of sort ~S cannot unify with a term of sort *S* or subsort of *S*, does not work and it makes the system *incomplete*. The example illustrating this point is given in [Jay 87].

Sorts are unary predicates. A possibility of extending sorts to two-place predicates, where there is a functional relationship between the two arguments, can be explored.

As for special relations, our proof procedure can handle only equality. This can be extended to include other relations, as it has been done for non-clausal paramodula-

tion and non-clausal E-resolution [MaW 86]. The possibility of generalizing the heuristics developed for RUE resolution [DiH 86] for efficiency to the present work can also be investigated.

Although many-sorted, equality-based approaches reduce the search space considerably, theorem provers cannot prove interesting theorems in *realistic domains* that include special relations unless more domain dependent information is captured and efficient strategies for guiding the theorem prover are used. Our hope is that the MWRUE resolution procedure will prove to be a reasonable starting point to investigate *Knowledge-based theorem provers.* In the simplest case it would be possible for humans to guide the prover much more easily than was possible earlier.

APPENDIX I

Theorem 4.7 Completeness of NCRUE Resolution (Open form, Ground case)

If U is an E-unsatisfiable set of ground wffs, there exists an NCRUE-NCNRF deduction of *FALSE* from U.

Proof:

Let k be the total number of binary connectives in U. Our proof is by induction on k.

For example, k for the following set of wffs U,

U_1: $(P \leftarrow\!\!\rightarrow Q) \vee R$

U_2: $\sim P \ \& \ Q \ \& \sim R$

is 4.

If $k = 0$, then U consists of only ground unit wffs. By Theorem 4.6, there is a unit NCRUE refutation of U that is factor free.

Now suppose our theorem holds for U ground and $k \leq n$. We show that it holds for $k = n + 1$.

Suppose ground U has $n + 1$ binary connectives. Consider a wff U_1 of U having a binary connective b. Depending on what connective this b is, we will have

different cases. Let the sub-wff containing this binary con-nective be $F_1\ b\ F_2$ in U_1.

Case(i): $F_1 \vee F_2$

Define U' as $U - \{U_1\}$ and U_1' as $U_1 - \{F_2\}$, and U_1'' as $U_1 - \{F_1\}$. Since U is E-unsatisfiable, so are $U' \cup \{U_1'\}$ and $U' \cup \{U_1''\}$. Since both of these have n or fewer binary connectives, they both have NCRUE-NCNRF refutations.

Now consider the refutation R_1 of $U' \cup \{U_1'\}$. Modify R_1 by adding back the sub-wff F_2 to the wff U_1'. This may give an NCRUE-NCNRF refutation of U in which case we derive a *FALSE* without the participation of F_2. Oth-erwise, this will give an NCRUE derivation of F_2 from U. By appending to this derivation the portion of the refuta-tion of $U' \cup \{U_1''\}$ where F_2 is refuted, we get an NCRUE-NCNRF refutation of ground U, which has n + 1 binary connectives.

We note that if $U_1' \cup \{F_2\}$ is used m times as an input wff in R_1, then the residue in R_1 is F_2 taken m times, $F_2 \vee F_2 \vee \ldots \vee F_2$, which is merged to the sub wff F_2. Hence, the refutation we have constructed to refute U is not necessarily factor free.

Case(ii): $F_1\ \&\ F_2$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_2\}$ and U_1'' as $U_1 - \{F_1\}$. Since U is E-unsatisfiable, either of $U' \cup \{U_1'\}$ or $U' \cup \{U_1''\}$ is E-unsatisfiable, or both, o

at least $U' \cup \{U_1'\}$ will give a derivation of $\sim F_2$.

Let us assume that $U' \cup \{U_1'\}$ is E-unsatisfiable Since it has n or fewer binary connectives, there is a NCRUE-NCNRF refutation. This will give us an NCRUE-NCNR refutation of U as well, because when we reduce F_1 i the sub-wff $(F_1 \& F_2)$ to *FALSE*, irrespective of F_2 th whole sub-wff reduces to *FALSE*.

If $U' \cup \{U_1''\}$ is E-unsatisfiable, then it is a dua case of the above.

If $U' \cup \{U_1'\}$ gives a derivation of $\sim F_2$, the resolve it with the sub-wff $(F_1 \& F_2)$ when it is derive in R with the sub-wff F_2 added back, to give an NCRUE NCNRF refutation of U.

Hence, we have got an NCRUE-NCNRF refutation of U, which has $n + 1$ binary connectives.

Case(iii): $(a)\ F_1 \text{--}> F_2$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_1 \text{--}> F_2\} \cup \{\sim F_1\}$ and U_1'' as $U_1 - \{F_1\}$.

The proof is same as that for Case(i).

$(b)\ F_2 \text{--}> F_1$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_2\}$ and U_1'' as. $U_1 - \{F_2 \text{--}> F_1\} \cup \{\sim F_2\}$.

Now the proof goes along the same lines of Case(i).

In both (iii) (a) and (b) the proof simply involv

rewriting $F_1 \longrightarrow F_2$ and $F_2 \longrightarrow F_1$ as $\sim F_1 \vee F_2$ and $\sim F_2 \vee F_1$ respectively.

Case(iv): $F_1 \longleftrightarrow F_2$

Define U' as $U - \{U_1\}$, U_1' as $U_1 - \{F_1 \longleftrightarrow F_2\} \cup \{F_1 \longrightarrow F_2\}$ and U_1'' as $U_1 - \{F_1 \longleftrightarrow F_2\} \cup \{F_2 \longrightarrow F_1\}$.

Since U is E-unsatisfiable, either of $U' \cup \{U_1'\}$ or $U' \cup \{U_1''\}$ or both are E-unsatisfiable.

Let us assume that $U' \cup \{U_1'\}$ is E-unsatisfiable. But, it has $n + 1$ binary connectives. So, by the Case(iii)(a), there is an NCRUE-NCNRF refutation. This will give us an NCRUE-NCNRF refutation of U as well.

If $U' \cup \{U_1''\}$ is E-unsatisfiable, then by Case(iii)(b), there is an NCRUE-NCNRF refutation. This will give then an NCRUE-NCNRF refutation of U.

Hence we have got an NCRUE-NCNRF refutation of U, which has $n + 1$ binary connectives.

Thus, we have proved the existence of an NCRUE-NCNRF refutation of U, which has $n + 1$ binary connectives in all possible cases.

This completes the induction and proves that if U is ground and E-unsatisfiable, it has an NCRUE-NCNRF refutation. $\square$

REFERENCES

[BoM 79] Boyer, R.S., and Moore, J.S. *A Computational Logic*, Academic Press, Orlando, Fla., 1979.

[Cha 79] De Champeaux, D. "Sub-problem finder and Instance checker: Two cooperating preprocessors for Theorem Provers", *IJCAI* 6, 1979, pp.191-196.

[ChL 73] Chang, C.L., and Lee, R.C. *Symbolic Logic and Mechanical Theorem Proving*, Academic Press, Orlando, Fla., 1973.

[DaP 60] Davis, M., and Putnam, H. "A Computing Procedure for Quantification Theory", *JACM* 7(3), 1960, pp.201-215.

[Dig 83] Digricoli, V.J. *Resolution by Unification and Equality*, Doctoral Dissertation, New York University, New York, 1983.

[DiH 86] Digricoli, V.J., and Harrison, M.C. "Equality-Based Binary Resolution", *JACM* 33(2), 1986, pp.253-289.

[Gen 34] Gentzen, G. "Untersuchungen über das Logische Schliessen", *Mathematische Zeitschrift* 39, 1934, pp.176-210, 405-431.

[MaW 80] Manna, Z., and Waldinger, R. "A Deductive Approach to Program Synthesis", *TOPLAS* 2(1), 1980, pp.90-121.

[MaW 82] Manna, Z., and Waldinger, R. *Special Relations in Program-Synthetic Deduction*, Tech. Report, Comp. Sci. Dept., Stanford Univ., Stanford, California, and Artificial Intelligence Center, SRI International, Menlo Park, California, March 1982.

[MaW 86] Manna, Z., and Waldinger, R. "Special Relations in Automated Deduction", *JACM* 33(1), 1986, pp.1-59.

[Mor 69] Morris, J.B. "E-resolution: An extension of Resolution to include the Equality relation", *Proc. of IJCAI*, Washington D.C., 1969, pp.287-294.

[Mur 82] Murray, N.V. "Completely Non-Clausal Theorem Proving", *AI* 18(1), 1982, pp.67-85.

[Pel 86] Pelletier, F.J. "Seventy-Five Problems for testing Automatic Theorem Provers", *J. of Automated Reasoning* 2, 1986, pp. 191-216.

[Qui 61] Quine, W.V. *Methods of Logic*, Holt, Rinehart and Winston Inc., New York, 1961.

[Rob 65] Robinson, J.A. "A Machine-oriented Logic Based on Resolution Principle", *JACM* 12(1), 1965, pp.23-41.

[Sch 38] Schmidt, A. "Über deduktive Theorien mit Mehrerer Sorten von Grunddingen", *Mathematische Annalen* 115, 1938, pp.485-506.

[Sch 85] Schmidt-Schauss, M. *Mechanical Generation of Sorts in Clause Sets*, SEKI Memo, Univ. of Kaiserslautern, WG, 1985.

[Sko 20] Skolem, Thoralf, "Logisch-Kambinatorische untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theorem über dichte Mengen", *Videnskopsselskapits skifter I. Mathematik-naturvidenskabelig klasse*, No. 4, 1920.

(Translated as: Logico-Combinatorial investigations in the satisfiability or provability of mathematical propositions: A simplified proof of a theorem by L.Löwenheim and generalizations of the theorem [Hei 67])

[Sko 28] Skolem, Thoralf, "Über die mathematische Logik", *Norsk matematisk tideskrift* 10, 1928, pp.125-142. (Translated as: On Mathematical Logic [Hei 67])

[Sti 85] Stickel, M.E. "A Non-Clausal Connection-Graph Resolution Theorem Proving Program", *Proc. Nat Conf. on AI*, Pittsburgh, 1985 and *American Association for AI*, Menlo Park, 1982, pp.229-233.

[Tar 51] Tarski, Alfred *A Decision Method for Elementary Algebra and Geometry*, Berkeley, 1951.

[Wal 82] Walther, Christoph *A Many-Sorted Calculus based on Resolution and Paramodulation*, SEKI Memo, Univ. Karlsruhe, WG, 1982.

[Wan 52] Wang, H. "Logic of Many-Sorted Theories", *JSL* 17(2), 1952.

[WOL 84] Wos, L., Overbeek, R., Lusk, E., and Boyle, J. *Automated Reasoning – Introduction and Applications*, PHI, NJ, 1984.

[WoR 69] Wos, L.A., and Robinson, G.A. "Paramodulation and Theorem Proving in First-Order Theories with Equality", *Machine Intelligence* 4, 1969, pp.135-150.